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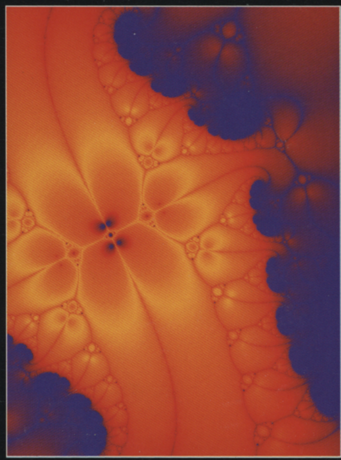
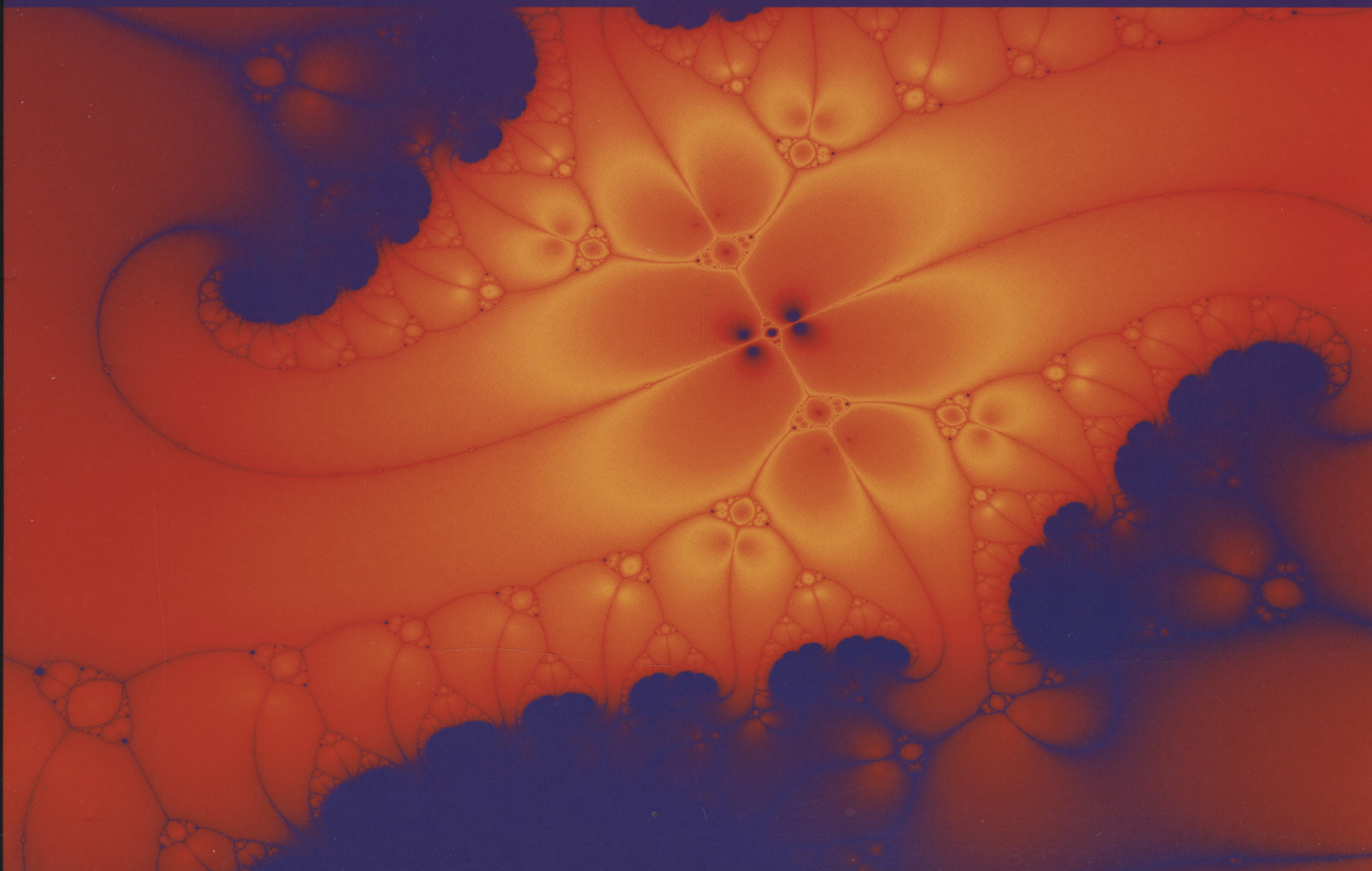
M338 Topology



C3

Unit C3 Sequences



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Sequences

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1.1

Contents

Introduction

In *Unit C2*, you met the notion of a *compact* set in a topological space and you saw that real-valued continuous functions defined on compact sets satisfy the General Extreme Value Theorem — that is, they are bounded and attain their bounds. For Euclidean spaces, you saw that there is a simple characterization of compact sets — *a set is compact if and only if it is closed and bounded* — which makes it straightforward to use the General Extreme Value Theorem.

Although the General Extreme Value Theorem applies to compact subsets of any metric space, in spaces such as $C[0,1]$, we currently have no convenient characterization of compact sets. In this unit, we address this issue. Our approach is to return to the notion of a sequence and investigate what it means for a sequence to be convergent in a topological space and in a metric space. We find that continuity in metric spaces can be characterized in terms of convergent sequences, but that this is not the case in a general topological space.

We also look in some detail at the convergence of sequences of continuous functions, and ask when the limit of a sequence of continuous functions is also continuous. We prove a theorem, the Uniform Convergence Theorem, that gives sufficient conditions to ensure that this happens.

In the last part of this unit, we show that, in metric spaces, there is a useful characterization of compact sets in terms of sequences. In particular, this gives a relatively simple technique that we can use to verify whether a given subset of $C[0,1]$ is compact, thereby allowing us to use theorems about compact sets from *Unit C2* in this setting.

Study guide

In Section 1, *Sequences in topological spaces*, we generalize the notion of a convergent sequence to topological spaces. This is an important section, as its results underlie everything else in this unit.

In Section 2, *Sequences in metric spaces*, we show that, in metric spaces, convergent sequences possess useful properties beyond those that they possess in a general topological space.

In Section 3, *Convergence of sequences of functions*, we investigate the convergence of sequences of functions. In particular, we discuss the distinction between *pointwise* convergence and *uniform* convergence.

In Section 4, *Sequences and compact sets in metric spaces*, we investigate the relationship between convergent sequences and compact sets, and introduce the notion of *sequential compactness*. You should make sure that you understand the statements of the main results of this section but, if you are short of time, you may wish to omit the proofs of these results. In particular, you may wish to omit Subsection 4.2.

Finally, in Section 5, *Compact subsets of* $C[0,1]$, we see how the equivalence between sequential compactness and compactness gives us a test for checking whether a given subset of $C[0,1]$ is compact. If you are short of time, concentrate on understanding how to use this test to verify whether a given subset of $(C[0,1], d_{\max})$ is compact, and if necessary omit Subsection 5.2.

There is no software associated with this unit.

1 Sequences in topological spaces

After working through this section, you should be able to:

- give examples of sequences in topological spaces;
- define *convergence* for sequences in a topological space;
- determine whether a given sequence in a topological space converges.

We begin our discussion of sequences in general topological spaces by recalling the definition of a *real* sequence. By generalizing the definition of convergence for real-valued sequences, we obtain a definition of convergence for sequences in topological spaces. We then look at properties of sequences in topological spaces, and investigate how they relate to open and closed sets. As you will see, some sequences in topological spaces have strange properties.

1.1 Sequences

In *Unit A1*, we defined a real sequence as follows.

Unit A1, Subsection 1.3.

Definition

A **real sequence** is an unending ordered list of real numbers

$$a_1, a_2, a_3, \ldots .$$

The real number a_n ($n \in \mathbb{N}$) is the ***n*th term** of the sequence, and the whole sequence is denoted by (a_n), $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n\in\mathbb{N}}$.

In *Unit A1*, we observed that a real sequence can be thought of as a function $a\colon \mathbb{N} \to \mathbb{R}$, given by $n \mapsto a_n$. Note that the only role played by $\mathbb{R}$ here is as the codomain of the function $a\colon \mathbb{N} \to \mathbb{R}$; the structure of $\mathbb{R}$ becomes relevant only when convergence is considered. Since the codomain of a function is simply a set, the following definition is a natural generalization.

Definition

Let X be a set. A **sequence** in X is an unending ordered list of elements of X

$$a_1, a_2, a_3, \ldots .$$

The element a_n of X ($n \in \mathbb{N}$) is the ***n*th term** of the sequence, and the sequence is denoted by (a_n), $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n\in\mathbb{N}}$.

Remarks

(i) This notion of a sequence is very general, and does not require that we impose any additional structure (such as a topology) on the set X.

(ii) The order of the terms of a sequence (a_n) is important. For example, the sequence $(-1, 1, -1, 1, \ldots)$ is different from the sequence $(1, -1, 1, -1, \ldots)$, and both sequences differ from the *set* $\{-1, 1, -1, 1, \ldots\} = \{-1, 1\}$. However, it is sometimes useful to consider the **set of the sequence** $\{a_n : n \in \mathbb{N}\}$.

Example 1.1

Examples of sequences in the plane include the sequence

$$(a_n) \text{ where } a_n = (\cos n, \sin n)$$

consisting of certain points on the unit circle, and the sequence

$$(b_n) \text{ where } b_n = \left(\tfrac{1}{n}, n\right)$$

consisting of certain points on the graph of $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$ (see Figure 1.1). ■

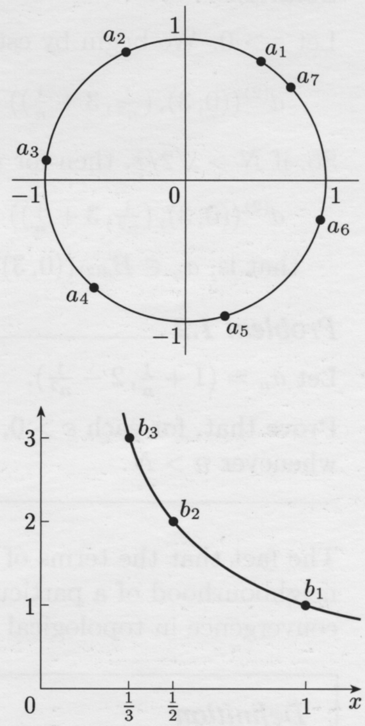


Figure 1.1

Problem 1.1

Give an example of a sequence in X, when:

(a) X is the Cantor space $\mathbf{C}$ consisting of all infinite sequences of 0s and 1s;

(b) X is $C[0, 1]$.

1.2 Convergent sequences

In *Unit A1*, we defined a *convergent* sequence of real numbers.

Unit A1, Subsection 1.3.

> ***Definition***
>
> A real sequence (a_n) **converges** to $l \in \mathbb{R}$ if, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that
>
> $$|a_n - l| < \varepsilon \quad \text{whenever } n > N$$
>
> — that is, a_n is in the open interval $(l - \varepsilon, l + \varepsilon)$ whenever $n > N$.
>
> We write $a_n \to l$ as $n \to \infty$, or simply $a_n \to l$.

Our immediate objective is to generalize this definition to topological spaces.

Consider, for example, the sequence (a_n) in the plane whose terms are the points

$$a_n = \left(1 + \tfrac{1}{n}, 2 - \tfrac{1}{n^2}\right).$$

As n increases, the terms of the sequence get closer and closer to the point $(1, 2) \in \mathbb{R}^2$ (see Figure 1.2). It seems that this sequence should converge with limit $(1, 2)$. The key to formulating a successful definition of convergence is the observation that, for each $\varepsilon > 0$, the terms of the sequence (a_n) eventually lie in the ball $B_{d^{(2)}}((1, 2), \varepsilon)$ — that is, there is an $N \in \mathbb{N}$ such that $a_n \in B_{d^{(2)}}((1, 2), \varepsilon)$ whenever $n > N$.

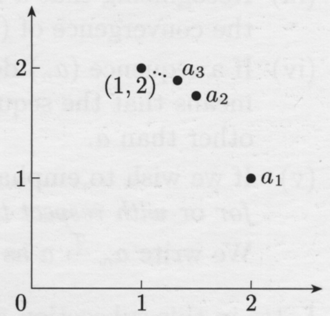


Figure 1.2

Worked problem 1.1

Let $a_n = \left(\frac{1}{n^2}, 3 + \frac{1}{n}\right)$. Prove that, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n \in B_{d^{(2)}}((0, 3), \varepsilon)$ whenever $n > N$.

Solution

Let $\varepsilon > 0$. We begin by estimating $d^{(2)}((0,3), a_n)$ for $n \in \mathbb{N}$. We have

$$d^{(2)}((0,3), (\tfrac{1}{n^2}, 3 + \tfrac{1}{n})) = \sqrt{(\tfrac{1}{n^2})^2 + (\tfrac{1}{n})^2} \leq \sqrt{(\tfrac{1}{n})^2 + (\tfrac{1}{n})^2} = \tfrac{\sqrt{2}}{n}.$$

So, if $N > \sqrt{2}/\varepsilon$, then for $n > N$,

$$d^{(2)}((0,3), (\tfrac{1}{n^2}, 3 + \tfrac{1}{n})) \leq \tfrac{\sqrt{2}}{n} < \varepsilon$$

— that is, $a_n \in B_{d^{(2)}}((0,3), \varepsilon)$ whenever $n > N$. ■

Problem 1.2

Let $a_n = (1 + \frac{1}{n}, 2 - \frac{1}{n^2})$.

Prove that, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n \in B_{d^{(2)}}((1,2), \varepsilon)$ whenever $n > N$.

The fact that the terms of a sequence (a_n) eventually lie in any given neighbourhood of a particular point is the essence of the definition of convergence in topological spaces.

Definition

Let $(X, \mathcal{T})$ be a topological space. A sequence (a_n) in X **converges** to $a \in X$ if, for each neighbourhood U of a, there is an $N \in \mathbb{N}$ such that

$$a_n \in U \quad \text{whenever } n > N.$$

We write $a_n \to a$ as $n \to \infty$, or simply $a_n \to a$.

We say that the sequence (a_n) is **convergent** in X with **limit** a.

A sequence that does not converge to any point in X is **divergent**.

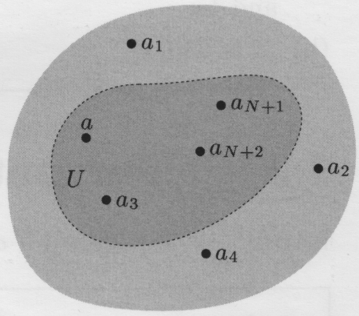


Figure 1.3

Remarks

(i) The point a *must* be an element of the set X.

(ii) A useful way to remember this definition is to rephrase it as

a_n converges to a if each neighbourhood of a contains all but finitely many terms of the sequence.

(iii) Recognizing that a is the *limit* of the sequence, we sometimes denote the convergence of (a_n) to a by $\lim_{n\to\infty} a_n = a$.

(iv) If a sequence (a_n) does not converge to a, then we write $a_n \not\to a$. This means that the sequence is divergent or it converges to some point other than a.

(v) If we wish to emphasize the topology, we say that (a_n) converges to a *for* or *with respect to* the topology $\mathcal{T}$, or that (a_n) is $\mathcal{T}$*-convergent*. We write $a_n \xrightarrow{\mathcal{T}} a$ as $n \to \infty$, or simply $a_n \xrightarrow{\mathcal{T}} a$.

Similar remarks apply to divergence.

Later in this subsection we verify that this definition of convergence coincides with the notion of convergence in *Unit A1* when the topology on $\mathbb{R}$ is the Euclidean one. First we look at some examples of sequences in other topological spaces. We begin by considering convergence for the discrete and indiscrete topologies.

Worked problem 1.2 *Discrete topology*

Let X be a non-empty set, and let $\mathcal{T}$ be the discrete topology on X. Let (a_n) be a sequence in X and let $a \in X$. Show that $a_n \to a$ as $n \to \infty$ if and only if (a_n) is eventually equal to a.

For the discrete topology, every subset of X is open.

Solution

Suppose that (a_n) is eventually equal to a — that is, there is an $N \in \mathbb{N}$ such that $a_n = a$ for all $n > N$. Now each neighbourhood U of a contains the point a, and so $a_n \in U$ for $n > N$. Thus $a_n \to a$ as $n \to \infty$.

Conversely, suppose that (a_n) is not eventually equal to a. Now $\{a\}$ is itself a neighbourhood of a for the discrete topology. Since there is no $N \in \mathbb{N}$ such that $a_n \in \{a\}$ for all $n > N$, it follows that $a_n \not\to a$ as $n \to \infty$. ■

Worked problem 1.3 *Indiscrete topology*

Let X be a non-empty set and let $\mathcal{T}$ be the indiscrete topology on X. Let (a_n) be a sequence in X and let $a \in X$. Show that $a_n \to a$ as $n \to \infty$.

For the indiscrete topology, the only open sets are X and $\varnothing$.

Solution

The only neighbourhood of a is the whole space X, and all terms of the sequence (a_n) belong to X. So $a_n \to a$ as $n \to \infty$. ■

We have shown the following.

> For the discrete topology on a set X, a sequence (a_n) in X converges if and only if (a_n) is *eventually constant.*
>
> For the indiscrete topology on a set X, each sequence converges to *every* point of X.

In general, the convergence status of a sequence depends on the topology used. For, if X contains at least two points and (a_n) is a sequence in X that is *not* eventually constant, then (a_n) does not converge for the discrete topology but it does converge for the indiscrete topology (to each point of the space). It is particularly surprising that, in some topological spaces, a sequence can converge to more than one point!

Worked problem 1.4 *The either-or topology*

Let $X = [-1, 1]$ and let

$$\mathcal{T} = \{U \subseteq X : 0 \notin U\} \cup \{U \subseteq X : (-1, 1) \subseteq U\}.$$

This either-or topology on X was introduced in *Unit A3*, Problem 3.5.

(a) Let (a_n) be a sequence in X with $|a_n| < 1$ for all $n \in \mathbb{N}$. Show that $a_n \to 0$ as $n \to \infty$.

(b) Let $a_n = \frac{1}{2} + \frac{1}{n+1}$ for each $n \in \mathbb{N}$. Show that $a_n \not\to \frac{1}{2}$.

Solution

(a) Let (a_n) be a sequence in X with $|a_n| < 1$ for all $n \in \mathbb{N}$.

Let U be a neighbourhood of 0. Then, from the definition of $\mathcal{T}$, U contains the open interval $(-1, 1)$. Since $|a_n| < 1$ for all $n \in \mathbb{N}$, it follows that $a_n \in U$ for all $n \in \mathbb{N}$.

Thus $a_n \to 0$ as $n \to \infty$.

(b) In order to show that $(\frac{1}{2} + \frac{1}{n+1})_{n=1}^{\infty}$ does not converge to $\frac{1}{2}$, we must find a neighbourhood U of $\frac{1}{2}$ such that, for each $N \in \mathbb{N}$, there is an $n > N$ with $a_n \notin U$.

Let $U = \{\frac{1}{2}\}$. This is an open set that contains $\frac{1}{2}$, and so U is a neighbourhood of $\frac{1}{2}$. Moreover, $a_n \notin U$ for all $n \in \mathbb{N}$. We conclude that (a_n) does not converge to $\frac{1}{2}$. ■

This worked problem confirms that the convergence of a sequence depends upon the choice of topology: for the Euclidean topology on $[-1, 1]$, the sequence $(\frac{1}{2} + \frac{1}{n+1})$ does converge to $\frac{1}{2}$.

Problem 1.3

Let X be an infinite set and let $\mathcal{T}$ be the co-countable topology on X. Show that if a sequence (a_n) converges to $a \in X$, then (a_n) is eventually constant.

Recall that $U \in \mathcal{T}$ if either $U = \varnothing$ or U^c is countable.

Problem 1.4

Let X be an infinite set and let $a \in X$. Let $\mathcal{T}_a$ be the a-deleted-point topology on X.

Recall that $U \in \mathcal{T}_a$ if either $U = X$ or $a \notin U$.

(a) Show that each sequence in X converges to a.

(b) Let $p \in X$ be distinct from a. Show that the constant sequence $(p, p, p, \ldots)$ converges to both a and p, but does not converge to any other point in X.

We now verify that the notion of topological convergence generalizes the definition of convergence given in *Unit A1*. To do this, we characterize convergence for a topology defined by a metric in terms of the metric. This immediately gives the required result for convergence in $\mathbb{R}$ for the Euclidean topology.

> ***Theorem 1.1***
>
> Let (X, d) be a metric space, let $\mathcal{T}(d)$ be the topology generated by the metric d, let (a_n) be a sequence in X and let $a \in X$.
>
> Then $a_n \to a$ if and only if $(d(a_n, a))$ is a null sequence.

Recall that $U \in \mathcal{T}(d)$ if and only if U is d-open.

Recall that a null sequence is one that converges to 0 for the Euclidean topology on $\mathbb{R}$.

Proof

Suppose first that $a_n \to a$ as $n \to \infty$; we must show that $(d(a_n, a))$ is a null sequence.

Let $\varepsilon > 0$. Since $a_n \to a$ and $B_d(a, \varepsilon)$ is a neighbourhood of a, there is an $N \in \mathbb{N}$ such that $a_n \in B_d(a, \varepsilon)$ for all $n > N$. Thus

$$0 \le d(a_n, a) < \varepsilon \quad \text{for all } n > N.$$

Since ε is arbitrary, $(d(a_n, a))$ is a null sequence.

Conversely, suppose that $(d(a_n, a))$ is a null sequence; we must show that $a_n \to a$ as $n \to \infty$. Let U be a neighbourhood of a. Then, since U is d-open, there is an $\varepsilon > 0$ such that

$$B_d(a, \varepsilon) \subseteq U.$$

Since $(d(a_n, a))$ is a null sequence, there is an $N \in \mathbb{N}$ such that $0 \le d(a_n, a) < \varepsilon$ for all $n > N$. Thus

$$a_n \in B_d(a, \varepsilon) \subseteq U \quad \text{for all } n > N.$$

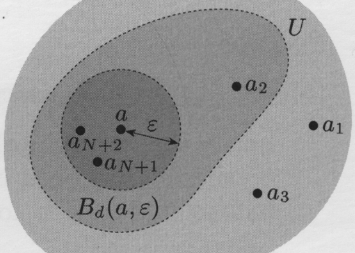


Figure 1.4

Since U is an arbitrary neighbourhood of a, we conclude that $a_n \to a$. ■

Applying Theorem 1.1 to the Euclidean topology on $\mathbb{R}$, we immediately deduce the following corollary.

Corollary 1.2

Let $\mathbb{R}$ possess the Euclidean topology, let (a_n) be a real sequence and let $a \in \mathbb{R}$. Then $a_n \to a$ as $n \to \infty$ if and only if, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$.

With this reassurance that our new definition of convergence agrees with the usual notion of convergence on the real line, we now analyse its consequences.

1.3 Convergence is a topological invariant

We have already seen that a convergent sequence can converge to more than one point, so we should not expect there to be many general properties of convergent sequences. Moreover, we should not expect to be able to obtain results such as the Sum and Product Rules for sequences that we had in *Unit A1*, since we may have no obvious way of 'adding' or 'multiplying' two sequences in a general topological space.

There are, however, some useful results about sequences in general topological spaces. In particular, we show that the convergence of sequences is a topological invariant: if two topological spaces are homeomorphic, then they have corresponding convergent sequences.

In fact, if (a_n) is a convergent sequence, then any *continuous* image of it is also convergent.

Theorem 1.3

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let (a_n) be a sequence in X with $a_n \xrightarrow{\mathcal{T}_X} a$ as $n \to \infty$. If $f\colon X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous, then $f(a_n) \xrightarrow{\mathcal{T}_Y} f(a)$ as $n \to \infty$.

Proof Let $U \subseteq Y$ be any neighbourhood of the point $f(a)$ — that is, U is an open set in $\mathcal{T}_Y$ containing $f(a)$. Since f is continuous, $f^{-1}(U)$ is an open set in $\mathcal{T}_X$. Moreover, it contains the point a — that is, $f^{-1}(U)$ is a neighbourhood of a.

Since $a_n \xrightarrow{\mathcal{T}_X} a$, there is an $N \in \mathbb{N}$ such that $a_n \in f^{-1}(U)$ for all $n > N$. But if $a_n \in f^{-1}(U)$, then $f(a_n) \in U$, and so $f(a_n) \in U$ for all $n > N$. Thus $f(a_n) \xrightarrow{\mathcal{T}_Y} f(a)$. ■

Remark

In *Unit A1*, we saw that, for the Euclidean topology, a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f(a_n) \to f(a)$ whenever $a_n \to a$. Unfortunately, in a general topological space, this may not be the case, since a convergent sequence need not have a unique limit.

If the function f in Theorem 1.3 is a homeomorphism, then it has an *inverse function* $f^{-1}: Y \to X$ that is $(\mathcal{T}_Y, \mathcal{T}_X)$-continuous. In this case, if $b_n \xrightarrow{\mathcal{T}_Y} b$, then the theorem applied to the function f^{-1} implies that $f^{-1}(b_n) \xrightarrow{\mathcal{T}_X} f^{-1}(b)$. This gives the following corollary — the convergence of sequences is a topological invariant.

Corollary 1.4

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f: X \to Y$ be a homeomorphism and let (a_n) be a sequence in X. Then $a_n \xrightarrow{\mathcal{T}_X} a$ if and only if $f(a_n) \xrightarrow{\mathcal{T}_Y} f(a)$.

Theorem 1.3 also enables us to prove the following result.

Theorem 1.5

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X, with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. If a sequence converges with respect to $\mathcal{T}_2$, then it converges to the same limit(s) with respect to $\mathcal{T}_1$.

Problem 1.5

Prove Theorem 1.5.

Hint Consider the identity map.

Problem 1.6

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X, with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Give an example to show that if a sequence converges with respect to $\mathcal{T}_1$, then it does not necessarily converge to the same limit(s) with respect to $\mathcal{T}_2$.

Hint Consider the discrete and indiscrete topologies on X.

1.4 Convergence in product spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$ formed from $\mathcal{T}_X$ and $\mathcal{T}_Y$. Recall that the collection of sets $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a base for this topology. Thus a set belongs to $\mathcal{T}_{X \times Y}$ if and only if it is the union of a family of sets from the base $\mathcal{B}$.

See *Unit A3*, Subsection 5.4, for a discussion of the product topology.

A sequence in $X \times Y$ may be written as $((x_n, y_n))$ — that is, for each $n \in \mathbb{N}$, the ordered pair (x_n, y_n) belongs to $X \times Y$, with $x_n \in X$ and $y_n \in Y$. For example, if $X = Y = \mathbb{R}$, then

$$(1, -1),\ (2, 0),\ (3, 1),\ \ldots,\ (n, n-2),\ \ldots$$

is a sequence in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Definition

Let X and Y be sets and let $((x_n, y_n))$ be a sequence in $X \times Y$. The sequences (x_n) in X and (y_n) in Y are the **component sequences** of the **product sequence** $((x_n, y_n))$.

The component sequences for the above example are $(1, 2, 3, \ldots, n, \ldots)$ and $(-1, 0, 1, \ldots, n-2, \ldots)$.

If you were asked to guess how the convergence of the product sequence $((x_n, y_n))$ relates to the convergence of the component sequences (x_n) and (y_n), your first instinct might be that the convergence of the former implies, and is implied by, the convergence of the latter. We now prove that this is true.

> **Theorem 1.6**
>
> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{T}_{X\times Y}$ be the product topology on $X \times Y$. Suppose that $(x, y) \in X \times Y$ and $((x_n, y_n))$ is a sequence in $X \times Y$. Then
>
> $$(x_n, y_n) \overset{\mathcal{T}_{X\times Y}}{\to} (x, y) \quad \text{if and only if} \quad x_n \overset{\mathcal{T}_X}{\to} x \text{ and } y_n \overset{\mathcal{T}_Y}{\to} y.$$

Proof Suppose first that $(x_n, y_n) \overset{\mathcal{T}_{X\times Y}}{\to} (x, y)$. Consider the projection function $p_X : X \times Y \to X$. We know from *Unit A3* that p_X is $(\mathcal{T}_{X\times Y}, \mathcal{T}_X)$-continuous, and so it follows from Theorem 1.3 that

Unit A3, Theorem 5.6.

$$p_X((x_n, y_n)) \overset{\mathcal{T}_X}{\to} p_X((x, y)) \text{ as } n \to \infty;$$

that is, $x_n \overset{\mathcal{T}_X}{\to} x$ as $n \to \infty$. Similarly, $y_n \overset{\mathcal{T}_Y}{\to} y$ as $n \to \infty$.

Conversely, suppose that $x_n \overset{\mathcal{T}_X}{\to} x$ and $y_n \overset{\mathcal{T}_Y}{\to} y$. Let $\mathcal{B}$ be the usual base of the product topology,

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

Let $W \in \mathcal{T}_{X\times Y}$ be a neighbourhood of (x, y). The definition of $\mathcal{T}_{X\times Y}$ implies the existence of a family $\{U_i \times V_i : i \in I\}$ in the base $\mathcal{B}$ such that W is the union over this family:

$$W = \bigcup_{i\in I}(U_i \times V_i).$$

Thus, since $(x, y) \in W$, there is an $i \in I$ for which $(x, y) \in U_i \times V_i$, and so $x \in U_i$ and $y \in V_i$.

Since $x_n \overset{\mathcal{T}_X}{\to} x$ and $y_n \overset{\mathcal{T}_Y}{\to} y$, there are $N, M \in \mathbb{N}$ such that $x_n \in U_i$ for all $n > N$ and $y_n \in V_i$ for all $n > M$. Hence, for all $n > \max\{N, M\}$, $(x_n, y_n) \in U_i \times V_i \subseteq W$. We conclude that $(x_n, y_n) \overset{\mathcal{T}_{X\times Y}}{\to} (x, y)$. ■

Remark

By induction, we can prove an analogous result for the product of k topological spaces:

Let $(X^1, \mathcal{T}^1)$, $(X^2, \mathcal{T}^2)$, ..., $(X^k, \mathcal{T}^k)$ be topological spaces, and let $\mathcal{W}$ be the product topology on the set $X^1 \times X^2 \times \cdots \times X^k$. A sequence $((x_n^1, x_n^2, \ldots, x_n^k))$ in the product set $X^1 \times X^2 \times \cdots \times X^k$ converges to $(x^1, x^2, \ldots, x^k)$ with respect to the product topology $\mathcal{W}$ if and only if each component sequence converges for its own topology: $x_n^j \overset{\mathcal{T}_j}{\to} x^j$ as $n \to \infty$, for $j = 1, 2, \ldots, k$.

We cannot avoid the double indexing: the superscript indicates which space the point belongs to, and the subscript is the sequence label.

As an application of Theorem 1.6, we look briefly at convergence in the Euclidean spaces $\mathbb{R}^k$. Since $\mathbb{R}^k$ is the product of k copies of $\mathbb{R}$, we immediately have the following result.

> **Theorem 1.7**
>
> For the Euclidean topologies, a sequence in $\mathbb{R}^k$ converges if and only if its component (real) sequences converge.

Worked problem 1.5

Determine whether each of the following sequences in $\mathbb{R}^2$ converges with respect to the Euclidean topology on $\mathbb{R}^2$. Write down the limit(s) of any sequence that converges.

(a) $(1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{8}), \ldots, (\frac{1}{n}, \frac{1}{2^n}), \ldots$

(b) $(1, 1), (\frac{1}{2}, -1), (\frac{1}{3}, 1), \ldots, (\frac{1}{n}, (-1)^{n+1}), \ldots$

Solution

(a) The component sequences, $(\frac{1}{n})$, and $(\frac{1}{2^n})$ both converge to 0 for the Euclidean topology on $\mathbb{R}$. Therefore the sequence converges to $(0, 0)$ for the product topology.

(b) The second component sequence is $((-1)^{n+1})$, which does not converge. Hence the sequence does not converge. ■

Problem 1.7

Determine whether each of the following sequences converges with respect to the Euclidean topologies on $\mathbb{R}^2$ or $\mathbb{R}^3$. Write down the limit(s) of any sequence that converges.

(a) $((1 - \frac{1}{n}, 1 + \frac{1}{n}))$

(b) $((\sin n\pi, \cos n\pi))$

(c) $((\frac{3^n}{n!}, \frac{(-1)^n}{n}, \frac{n-5}{n^2+2n+1}))$

Problem 1.8

Consider the sequence $((\frac{1}{n}, \frac{1}{n^2}))$ in $\mathbb{R}^2$, and let $\mathcal{T}$ be the product topology obtained by taking the product of the Euclidean topology with the discrete topology. Does the sequence converge for $\mathcal{T}$ and, if so, to what?

1.5 Convergence in Hausdorff spaces

We end this section by returning to a situation that we encountered earlier: in a topological space, a convergent sequence may have more than one limit. However, there is an important class of topological spaces for which the limit of any convergent sequence is unique — Hausdorff spaces. Recall that a topological space $(X, \mathcal{T})$ is *Hausdorff* if for all distinct $x, y \in X$, there are disjoint neighbourhoods of x and y.

You studied Hausdorff spaces in *Unit C2*, Section 3.

> **Theorem 1.8**
>
> Let $(X, \mathcal{T})$ be a Hausdorff space and let $a, b \in X$. If (a_n) is a sequence in X that converges to both a and b, then $a = b$.

Problem 1.9

Prove Theorem 1.8.

Hint Use a proof by contradiction — suppose that $a \neq b$ and consider disjoint neighbourhoods of a and b.

2 Sequences in metric spaces

After working through this section, you should be able to:

- ► define convergence for sequences in a metric space;
- ► determine whether a given sequence in a metric space converges;
- ► find the closure of a set in a metric space by using sequences.

In topological spaces, convergent sequences can have bad properties — for example, they need not have unique limits. In Hausdorff spaces, however, we have just seen that the limits *are* unique. Since our main examples of Hausdorff spaces are those for which the topology is generated by a metric, we now study the role that sequences play in metric spaces.

We show that there are many useful results for sequences in metric spaces. Indeed, one could say that sequences are most natural when considered in the context of metric spaces.

2.1 Limits of sequences in metric spaces

In Theorem 1.1, we saw that for topologies defined from metrics, there is a characterization of convergence in terms of the underlying metric.

> Let (X, d) be a metric space, let $\mathcal{T}(d)$ be the topology generated by the metric d, let (a_n) be a sequence in X and let $a \in X$.
>
> Then $a_n \to a$ if and only if $(d(a_n, a))$ is a null sequence.

This result underlies the following definition.

> ***Definition***
>
> Let (X, d) be a metric space. A sequence (a_n) in X **converges** to $a \in X$ if $(d(a_n, a))$ is a null sequence.
>
> We write $a_n \to a$ as $n \to \infty$, or simply $a_n \to a$.
>
> We say that the sequence (a_n) is **convergent** in X with **limit** a.
>
> A sequence that does not converge to any point in X is **divergent**.

Remarks

(i) Note that a must be a point in X. For example, if X is the interval $(0, 1)$ with the Euclidean metric, then the sequence $(\frac{1}{n})$ is not convergent, since the only possible limit is 0 which is not a point of X.

(ii) If we wish to emphasize the metric, we say that (a_n) converges *for* or *with respect to* the metric d, or that (a_n) is *d-convergent*. We write $a_n \overset{d}{\to} a$ as $n \to \infty$, or simply $a_n \overset{d}{\to} a$. Note that $a_n \overset{d}{\to} a$ if and only if $a_n \overset{\mathcal{T}(d)}{\to} a$.

Similar remarks apply to divergence.

(iii) In order to show that a sequence (a_n) converges to a for the metric d, it suffices to show that $d(a_n, a) \to 0$ as $n \to \infty$ — that is, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n \in B_d(a, \varepsilon)$ for all $n > N$.

This definition of convergence coincides with the definition of convergence for the topology defined by the metric, and so all our results concerning

convergence in topological spaces carry forward to this setting. In particular, since the topology defined by a metric is Hausdorff, Theorem 1.8 implies that in a metric space, each convergent sequence has a unique limit.

Unit C2, Theorem 3.1.

Proposition 2.1

In a metric space, each convergent sequence has a unique limit.

Problem 2.1

Let $\mathbf{C}$ be the Cantor space consisting of all infinite sequences of 0s and 1s. The Cantor distance $d_{\mathbf{C}}\colon \mathbf{C} \times \mathbf{C} \to \mathbb{R}$ is defined by

In *Unit A2*, Theorem 2.3, we showed that $d_{\mathbf{C}}$ is a metric on $\mathbf{C}$.

$$d_{\mathbf{C}}(\mathbf{x},\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ 2^{-n} & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ first differ at the } n\text{th term.} \end{cases}$$

Let $\mathbf{a}_n$ be the point in $\mathbf{C}$ with 1 for the first n terms and 0 for all the other terms. So

$$\mathbf{a}_1 = (1,0,0,\ldots),\ \mathbf{a}_2 = (1,1,0,0,\ldots),\ \ldots .$$

Show that $(\mathbf{a}_n)$ converges to $\mathbf{a} = (1,1,1,\ldots)$ in $\mathbf{C}$ for $d_{\mathbf{C}}$.

Problem 2.2

Let (X,d) be a metric space, and let e be the associated bounded metric on X given by

$$e(x,y) = \min\{1, d(x,y)\} \quad \text{for all } x,y \in X.$$

Let (a_n) be a sequence in X which converges to a with respect to the metric d. Show that (a_n) converges to a with respect to the metric e.

2.2 Convergence and closure in metric spaces

In *Unit A4* you studied closures of sets. In particular, you saw that a point x in a set X belongs to the closure of a subset $A \subseteq X$ if and only if each neighbourhood of x contains at least one point of A. Now that we have the concept of the convergence of sequences, we can give a new characterization of the closure of a subset of a metric space. This can be a useful tool for determining the closure of a set.

Theorem 2.2

Let (X,d) be a metric space and let A be a subset of X. A point a belongs to the closure $\mathrm{Cl}(A)$ of A if and only if there is a sequence (a_n) in A that converges to a.

Proof First we show that each closure point is the limit of a sequence in A.

Let $a \in \mathrm{Cl}(A)$. We construct a sequence of points $a_n \in A$ that converges to a, by choosing a sequence that lies in successively smaller open balls centred on a.

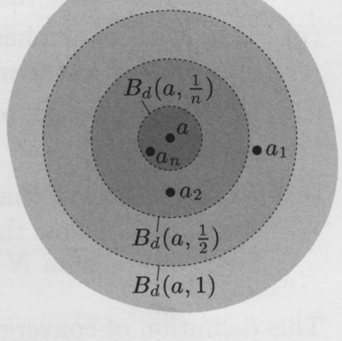


Figure 2.1

For each $n \in \mathbb{N}$, the ball $B_d(a, 1/n)$ meets A and so we may choose a point $a_n \in B_d(a,1/n) \cap A$. For this point, $d(a_n, a) < 1/n$. Hence, by Theorem 1.1, (a_n) is a sequence in A that converges to a.

We now show that if a is the limit of a sequence (a_n) in A, then a is a closure point of A. Let U be a neighbourhood of a. Since $a_n \to a$, there is an $N \in \mathbb{N}$ such that U contains a_n for all $n > N$. Since each $a_n \in A$, U contains points of A. Since U is an arbitrary neighbourhood of a, we conclude that a is a closure point of A. ■

Remarks

(i) Notice that the terms of the sequence (a_n) must belong to A, rather than simply to X.

(ii) Observe that the proof of the 'only if' statement holds for any topological space:

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. If $a \in X$ is the limit of a sequence in A, then $a \in \mathrm{Cl}(A)$.

However, the 'if' statement need not hold in every topological space — we ask you to verify this in the next problem.

Problem 2.3

Let $X = [0, 1]$ and let $\mathcal{T}$ denote the co-countable topology on X.

(a) Show that if $A = [0, \frac{1}{2}]$, then $\mathrm{Cl}(A) = [0, 1]$.

(b) Show that for each $a \in (\frac{1}{2}, 1]$, *no* sequence in A converges to a for $\mathcal{T}$.

Recall that $U \in \mathcal{T}$ if either $U = \varnothing$ or U^c is countable.

We conclude this section by showing that continuity and convergence are closely linked for a metric space.

Theorem 2.3

Let (X, d_X) and (Y, d_Y) be metric spaces. The function $f\colon X \to Y$ is (d_X, d_Y)-continuous if and only if

$$f(a_n) \xrightarrow{d_Y} f(a) \text{ whenever } (a_n) \text{ is a sequence in } X \text{ with } a_n \xrightarrow{d_X} a.$$

Proof Since $f\colon X \to Y$ is (d_X, d_Y)-continuous if and only if it is $(\mathcal{T}(d_X), \mathcal{T}(d_Y))$-continuous, and a sequence $b_n \xrightarrow{d} b$ if and only if $b_n \xrightarrow{\mathcal{T}(d)} b$, Theorem 1.3 implies that

if $f\colon X \to Y$ is (d_X, d_Y)-continuous and (a_n) is a sequence in X with $a_n \xrightarrow{\mathcal{T}_X} a$, then $f(a_n) \xrightarrow{\mathcal{T}_Y} f(a)$.

For the converse result, we use the characterization of continuity in terms of closed sets:

f is continuous if $f^{-1}(D)$ is closed in X for each closed set D in Y.

Unit A4, Subsection 1.4.

For each $a \in X$, suppose that $f(a_n) \to f(a)$ for each sequence (a_n) converging to a in X. Let D be a closed subset of Y and let $A = f^{-1}(D)$. To show that A is closed, it is sufficient to show that $\mathrm{Cl}(A) \subseteq A$.

Unit A4, Theorems 2.2 and 2.4.

If $\mathrm{Cl}(A) = \varnothing$, then there is nothing to prove. So suppose that $a \in \mathrm{Cl}(A)$. Then Theorem 2.2 implies that there is a sequence (a_n) in A with $a_n \xrightarrow{d_X} a$ as $n \to \infty$. Thus, for each $n \in \mathbb{N}$,

$$f(a_n) \in f(A) = f(f^{-1}(D)) \subseteq D,$$

and so $(f(a_n))$ is a sequence in D. By our hypothesis $(f(a_n))$ converges to $f(a)$, and so, by Theorem 2.2 again, $f(a) \in \mathrm{Cl}(D)$. Since D is closed, $\mathrm{Cl}(D) = D$ and so $f(a) \in D$. Thus $a \in f^{-1}(D) = A$ and hence $\mathrm{Cl}(A) \subseteq A$, so A is closed. Hence f is continuous, as required. ■

Unit A4, Theorem 2.4.

3 Convergence of sequences of functions

After working through this section, you should be able to:

- define *pointwise convergence* and *uniform convergence* for sequences of functions;
- determine whether a given sequence of functions converges pointwise;
- determine whether a given sequence of functions converges uniformly;
- find the limit of a given sequence of functions in $C[0,1]$ for $d_{\max}$.

In this section we study two possible ways of defining convergence for sequences of functions — *pointwise convergence* and *uniform convergence*.

Once we have these notions, we can ask whether, if the functions $f_n: X \to Y$ are continuous (with respect to some metrics), the limit function, if it exists, is necessarily continuous. We show that the notion of pointwise convergence does not guarantee this, but that uniform convergence does.

3.1 Pointwise convergence

We first discuss the simplest way of defining convergence for sequences of real-valued functions — pointwise convergence.

If A is some fixed set and $f_n: A \to \mathbb{R}$ for each $n \in \mathbb{N}$, then, for each $x \in A$, $(f_n(x))$ is a sequence of real numbers. Informally, pointwise convergence involves looking separately at each x in the set A, and investigating whether the corresponding real sequence $(f_n(x))$ converges.

Definition

Let A be a set.

A sequence (f_n) of functions $f_n: A \to \mathbb{R}$ **converges pointwise** to the function $f: A \to \mathbb{R}$ if the sequence of real numbers $(f_n(x))$ converges to $f(x)$, for each $x \in A$.

If a sequence (f_n) of functions converges pointwise to a function f, then f is the **pointwise limit** of the sequence (f_n).

We write $f_n \to f$ pointwise as $n \to \infty$, or simply $f_n \to f$ pointwise.

Remarks

(i) We do not require a topology to be defined on A. In particular, we need not know anything about the continuity of the functions f_n.

(ii) We know that the pointwise limit is unique (if it exists) because, for each $x \in A$, the sequence $(f_n(x))$ is a sequence of real numbers, and the convergence of a sequence of real numbers corresponds to convergence in $\mathbb{R}$ with respect to the Euclidean metric. In particular, limits (when they exist) are unique.

This notion of pointwise convergence can be extended to the situation when the codomain is a general topological space. In this case, there need not be a unique pointwise limit.

Worked problem 3.1

For each $n \in \mathbb{N}$, let $f_n\colon [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Find the pointwise limit of the sequence (f_n).

Solution

Since $x^n \to 0$ for all $x \in [0,1)$ but $x^n \to 1$ if $x = 1$, we need to consider the point $x = 1$ separately.

For a fixed $x \in [0,1)$, the sequence of values of the functions at x is $(f_n(x)) = (x^n)$, and this real sequence has limit 0 as $n \to \infty$. Thus, for $x \in [0,1)$, $f_n(x) \to 0$ as $n \to \infty$.

For $x = 1$ and all $n \in \mathbb{N}$, $f_n(1) = 1$, so $f_n(1) \to 1$ as $n \to \infty$.

Hence the pointwise limit of the sequence (f_n) is the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

■

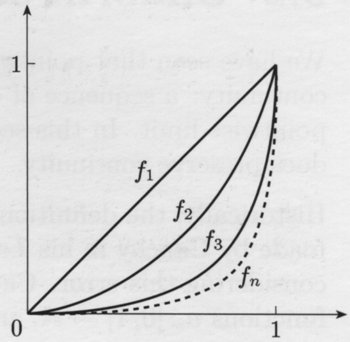


Figure 3.1 The functions f_n.

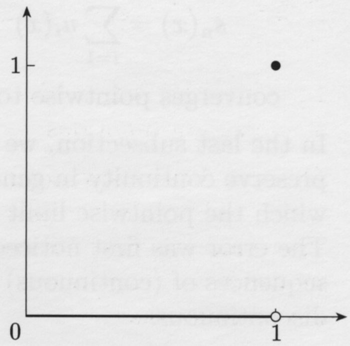


Figure 3.2 The function f.

Note that each of the functions f_n in Worked problem 3.1 is continuous (with respect to the Euclidean topologies), and yet the limit function is not continuous. This illustrates the fundamental disadvantage of pointwise convergence — it does not generally preserve 'nice' properties of functions. We shall see further examples of this shortly.

> The pointwise limit of a sequence of continuous functions need not be continuous.

Problem 3.1

For each $n \in \mathbb{N}$, let $f_n\colon [0,\infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2^n}, \\ -\frac{1}{2^n} & \text{if } \frac{1}{2^n} < x < \infty. \end{cases}$$

Find the pointwise limit of the sequence of functions (f_n).

Problem 3.2

For each $n \in \mathbb{N}$, let $f_n\colon [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2x & \text{if } 0 \le x \le \frac{1}{2n}, \\ -4n^2x + 4n & \text{if } \frac{1}{2n} < x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

(a) Sketch the graphs of f_1, f_2 and f_3.

(b) Find the pointwise limit f of the sequence (f_n), and calculate $\int_0^1 f(x)\,dx$.

(c) Find the limit of the real sequence $\left(\int_0^1 f_n(x)\,dx\right)$.

(d) What conclusion can be drawn about the relation between integration and pointwise convergence?

3.2 Uniform convergence

We have seen that pointwise convergence of functions does not preserve continuity: a sequence of continuous functions may have a discontinuous pointwise limit. In this section, we introduce a type of convergence that does preserve continuity.

Historically, the definition of uniform convergence arose out of a mistake made by Cauchy in his *Lectures on Analysis*, published in 1821. It is worth considering this error. Cauchy was considering an infinite *series* of functions $u_i\colon [0,1] \to \mathbb{R}$, and made the following assertion:

> if the functions u_i are all continuous and if the sequence of partial sums $s_n\colon [0,1] \to \mathbb{R}$ given by
>
> $$s_n(x) = \sum_{i=1}^{n} u_i(x)$$
>
> converges pointwise to a limit function s, then s is also continuous.

The function s_n is continuous, since it is a *finite* sum of continuous functions.

Here $s = \sum_{i=1}^{\infty} u_i$.

In the last subsection, we saw that pointwise convergence does not preserve continuity in general, and it is not difficult to find functions u_n for which the pointwise limit of the partial sums exists and is *not* continuous. The error was first noticed in 1826 by Abel, who presented examples of sequences of (continuous) partial sums whose pointwise limit is discontinuous.

Niels Henrik Abel (1802–29) investigated the convergence of sequences and series. He is mainly remembered for proving that, although polynomial equations of degrees 2, 3 and 4 can be solved by means of radicals (roots), there is no such general solution for polynomials of degree 5 or more.

Let us see where Cauchy's mistake lies. He wrote

$$s(x) = s_n(x) + \sum_{i=n+1}^{\infty} u_i(x),$$

and reasoned as follows. Since s_n converges to s as $n \to \infty$, given any $\varepsilon > 0$ we can choose an $N \in \mathbb{N}$, depending on ε, so that, for all $n > N$, the 'tail' of the series is less than ε:

$$\left| \sum_{i=n+1}^{\infty} u_i(x) \right| < \varepsilon.$$

Hence, he wrote, $s(x)$ is a finite sum of continuous functions plus a correction that is less than ε, and so s is continuous.

The problem is that, in general, N must depend on both ε *and* x, and so the whole chain of reasoning just given breaks down. But if the functions u_i are such that we *can* choose an N independently of x, then the result follows. In this case, the same N works *simultaneously* for all x.

To see what can go wrong, let us reconsider the convergence of the sequence (f_n) where $f_n\colon [0,1] \to \mathbb{R}$ is given by $f_n(x) = x^n$. We know that the limit function f is discontinuous with respect to the Euclidean topologies, and so whatever goes wrong with pointwise convergence should be discernible from this sequence.

We showed in Worked problem 3.1 that the limit function $f\colon [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Suppose that $0 < \varepsilon < 1$ and $x \in (0,1)$. For what value of n does x^n become less than ε? Solving $x^r = \varepsilon$ for r, we find that $r = \log \varepsilon / \log x$. As anticipated, $r = r(\varepsilon, x)$ depends on both x and ε. Thus, if $N(\varepsilon, x) \in \mathbb{N}$ is chosen so that $N(\varepsilon, x) \ge r(\varepsilon, x)$, then $x^n < \varepsilon$ for all $n > N(\varepsilon, x)$.

The crucial question is: is there an integer $N(\varepsilon)$, depending on ε but not x, that can serve for a 'uniform rate' of decrease of the x^n over the entire domain $[0,1]$?

$N(\varepsilon)$ needs to be at least as large as all of the numbers $N(\varepsilon, x)$.

The answer is *no*, for as x approaches 1 from below, $N(\varepsilon, x)$ grows without limit: that is,

$$\sup\{N(\varepsilon, x) : x \in [0,1]\} \geq \sup\{\log \varepsilon / \log x : x \in (0,1)\} = \infty.$$

The hope is that if we require this *not* to happen — that is, if the set $\{N(\varepsilon, x) : x \in [0,1]\}$ is bounded above for each $\varepsilon > 0$, then we obtain a 'uniform control', and the limit function of a sequence of continuous functions *is* continuous. This is the idea underlying the notion of *uniform convergence.*

Before we define uniform convergence, we introduce the following terminology.

Definition

Let A be a set. A function $f\colon A \to \mathbb{R}$ is **bounded** on A if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in A$.

This extends the definition of boundedness of a function, which you met in *Unit A1.*

We now give the definition of uniform convergence.

Definition

Let A be a set. A sequence (f_n) of functions $f_n\colon A \to \mathbb{R}$ **converges uniformly** on A to the function $f\colon A \to \mathbb{R}$ if there is an $N \in \mathbb{N} \cup \{0\}$ such that

(a) the function $f_n - f$ is bounded for each $n > N$,

and

(b) the sequence (M_n) defined by

$$M_n = \sup\{|f_{n+N}(x) - f(x)| : x \in A\}$$

is a null sequence.

Such a function f is the **uniform limit** of the sequence.

We write $f_n \to f$ uniformly as $n \to \infty$, or simply $f_n \to f$ uniformly.

Remarks

(i) The requirement that $f_n - f$ is bounded for sufficiently large n is imposed to ensure that M_n is finite for each $n \in \mathbb{N}$.

(ii) As with the definition of pointwise convergence, we place no topological assumptions on the domain of the functions.

(iii) We must always specify on which domain the sequence converges uniformly. A sequence might converge uniformly on A but not on some $B \supset A$. This is illustrated in Worked problem 3.2 and Problem 3.4.

(iv) There is a useful visual aid to understanding the meaning of uniform convergence when $A = [0,1]$. Given $\varepsilon > 0$, we can draw a 'sleeve' of width 2ε around f. If, for any $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that the graphs of all the functions f_n for $n > N(\varepsilon)$ fit inside this sleeve over all of A, then the convergence is uniform. If there is at least one ε for which we cannot find such a sleeve over all of A, then the convergence is not uniform.

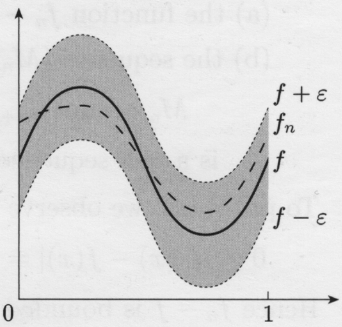


Figure 3.3

Before we look at some examples, we prove a lemma that is useful when trying to find the uniform limit of a sequence. It states that uniform convergence implies pointwise convergence.

Lemma 3.1

Let A be a set and let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$.

If the sequence (f_n) converges uniformly on A to $f : A \to \mathbb{R}$, then (f_n) also converges pointwise to f.

Proof Suppose that (f_n) converges uniformly to f. We show that, for each $x \in A$, the real sequence $(f_n(x))$ converges to $f(x)$.

Let $N \in \mathbb{N}$ be such that $f_n - f$ is a bounded function for all $n > N$ — that is, for all $n \in \mathbb{N}$,

$$M_n = \sup\{|f_{n+N}(x) - f(x)| : x \in A\} < \infty.$$

Then, for all $n \in \mathbb{N}$ and $x \in A$,

$$0 \leq |f_{n+N}(x) - f(x)| \leq M_n.$$

We know that (M_n) is a null sequence since $f_n \to f$ uniformly as $n \to \infty$. Hence, by the Squeeze Rule for sequences, $(f_{n+N}(x) - f(x))_{n \in \mathbb{N}}$ is a null sequence, and so $f_n(x) \to f(x)$ as $n \to \infty$. ■

Unit A1, Subsection 1.3.

This lemma tells us that if a sequence of functions converges uniformly to a function, then it also converges pointwise to the same function. Thus, in order to identify a possible uniform limit of a sequence of functions, it is enough to find the (unique) pointwise limit of the sequence. Having found this candidate for the uniform limit, we then have to show that it *is* the uniform limit. The following worked problem illustrates the method.

Worked problem 3.2

Let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = x/n.$$

Show that the sequence (f_n) converges uniformly on $[0,1]$, and find its uniform limit.

Solution

We begin by identifying the pointwise limit of the sequence. For each $x \in [0,1]$, $x/n \to 0$ as $n \to \infty$, and so the pointwise limit of the sequence (f_n) is the zero function $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$ for each $x \in [0,1]$.

Thus, by Lemma 3.1, our only candidate for the uniform limit of the sequence (f_n) is the function f. To show that it is the uniform limit, we must show that there is an $N \in \mathbb{N} \cup \{0\}$ such that:

(a) the function $f_n - f$ is bounded for each $n > N$;

(b) the sequence (M_n) defined by

$$M_n = \sup\{|f_{n+N}(x) - f(x)| : x \in [0,1]\}$$

is a null sequence.

To prove (a), we observe that, for $x \in [0,1]$ and $n \in \mathbb{N}$,

$$0 \leq |f_n(x) - f(x)| = |x/n - 0| = |x/n| \leq 1/n.$$

Hence $f_n - f$ is bounded for all $n \in \mathbb{N}$.

For (b), we note that, for all $n \in \mathbb{N}$,

$$0 \leq M_n = \sup\{|x/n - 0| : x \in [0,1]\} = 1/n.$$

Since $(\frac{1}{n})$ is a basic null sequence, it follows that (M_n) is a null sequence.

Thus $f_n \to f$ uniformly as $n \to \infty$. ■

Problem 3.3

Let $f_n\colon [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = 1 - (x/n)^2.$$

Show that the sequence (f_n) converges uniformly, and find its uniform limit.

Problem 3.4

Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = x/n.$$

Show that the sequence (f_n) does not converge uniformly.

We have seen in Worked problem 3.2 and Problem 3.4 that the functions given by $f_n(x) = x/n$ converge uniformly on $[0, 1]$, but do *not* converge uniformly on the whole of $\mathbb{R}$. Problem 3.4 also illustrates another point.

> Even when a sequence converges pointwise to a *continuous* function, the convergence need not be uniform.

Uniform convergence and continuity

We now show that uniform convergence preserves continuity — that is, *the uniform limit of a sequence of continuous functions is also continuous.*

Note that, in order to consider continuity, we need a topology or metric on the domain, as well as the Euclidean topology on the codomain $\mathbb{R}$.

> ***Theorem 3.2***
>
> Let (X, d) be a metric space and let $f_n\colon X \to \mathbb{R}$ be $(d, d^{(1)})$-continuous at $a \in X$ for each $n \in \mathbb{N}$.
>
> If the sequence (f_n) converges uniformly on X to a function $f\colon X \to \mathbb{R}$, then f is $(d, d^{(1)})$-continuous at a.

Proof Let $\varepsilon > 0$. We show that there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in B_d(a, \delta)$.

Using the Triangle Inequality twice, we find that

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|, \qquad (*) \end{aligned}$$

for each $x \in X$, and for all $n \in \mathbb{N}$. We now deal with each of these three terms.

Since $f_n \to f$ uniformly on X, we know that there exists an $N \in \mathbb{N} \cup \{0\}$ such that

$$|f_{n+N}(x) - f(x)| \le M_n$$

for all $x \in X$, where the sequence (M_n) is null.

Hence there is an $M \in \mathbb{N}$ such that

$$|f_M(x) - f(x)| < \tfrac{1}{3}\varepsilon,$$

for all $x \in X$. In particular,

$$|f_M(a) - f(a)| < \tfrac{1}{3}\varepsilon.$$

Moreover, f_M is continuous at a, and so there exists a $\delta > 0$ such that

$$|f_M(x) - f_M(a)| < \tfrac{1}{3}\varepsilon \quad \text{whenever } x \in B_d(a, \delta).$$

Combining these observations and using $(*)$, we have

$$|f(x) - f(a)| < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \quad \text{whenever } x \in B_d(a, \delta),$$

and so f is continuous at a. ■

This theorem is very useful, and we shall use it in the next unit.

Problem 3.5

For all $n \in \mathbb{N}$, let $f_n\colon [0, \infty) \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{\sin nx}{nx} & \text{if } x > 0. \end{cases}$$

Does this sequence converge pointwise? Does it converge uniformly? If it does converge, find its limit function.

Hint $\lim_{y \to 0} \frac{\sin y}{y} = 1$.

3.3 Sequences in $C[0, 1]$

We now relate the $d_{\max}$-convergence of sequences in $C[0, 1]$ to pointwise and uniform convergence.

Recall that $f \in C[0, 1]$ if $f\colon [0, 1] \to \mathbb{R}$ and f is continuous on $[0, 1]$ for the Euclidean topologies. The distance $d_{\max}$ between two functions f and g in $C[0, 1]$ is defined by

$$d_{\max}(f, g) = \max\{|g(x) - f(x)| : x \in [0, 1]\};$$

we saw in *Unit A2* that this defines a metric on $C[0, 1]$. *Unit A2*, Theorem 2.4.

Theorem 1.1 tells us that a sequence (f_n) in $C[0, 1]$ converges to f in $C[0, 1]$ for $d_{\max}$ if and only if $d_{\max}(f_n, f) \to 0$ as $n \to \infty$: that is, if and only if

$$\max\{|f(x) - f_n(x)| : x \in [0, 1]\} \to 0 \text{ as } n \to \infty.$$

The resemblance between this and part (b) of the definition of uniform convergence suggests the following result.

> ***Theorem 3.3 Uniform Convergence Theorem***
>
> Let (f_n) be a sequence of functions in $C[0, 1]$.
>
> If (f_n) converges uniformly on $[0, 1]$ to the function $f\colon [0, 1] \to \mathbb{R}$, then f is continuous and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$.
>
> Conversely, if $f \in C[0, 1]$ and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, then (f_n) converges uniformly to f on $[0, 1]$.

Proof Let (f_n) be a sequence in $C[0,1]$ that converges uniformly to the function $f\colon [0,1] \to \mathbb{R}$. By Theorem 3.2, f is continuous on $[0,1]$. It remains to show that $(d_{\max}(f_n, f))_{n\in\mathbb{N}}$ is a null sequence.

Since $f_n \to f$ uniformly, there is an $N \in \mathbb{N} \cup \{0\}$ such that $f_n - f$ is bounded for all $n > N$. In fact, since, for all $n \in \mathbb{N}$, $f_n - f$ is a continuous function on the compact set $[0,1]$, the General Extreme Value Theorem tells us that it is bounded for all $n \in \mathbb{N}$. Hence, by the definition of uniform convergence, the sequence (M_n) defined by

Unit C2, Theorem 4.3.

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$

is a null sequence.

Also, since $[0,1]$ is compact, the General Extreme Value Theorem tells us that $f_n - f$ attains its bounds, for all $n \in \mathbb{N}$. Thus

$$M_n = \max\{|f_n(x) - f(x)| : x \in [0,1]\} = d_{\max}(f_n, f).$$

Therefore $(d_{\max}(f_n, f))$ is a null sequence, as required.

Conversely, let (f_n) be a sequence of functions in $C[0,1]$ that converges to $f \in C[0,1]$ with respect to $d_{\max}$. Then $(d_{\max}(f_n, f))$ is a null sequence. Again we can deduce from the General Extreme Value Theorem that $f_n - f$ is bounded for all $n \in \mathbb{N}$ and that

$$\begin{aligned} M_n &= \sup\{|f_n(x) - f(x)| : x \in [0,1]\} \\ &= \max\{|f_n(x) - f(x)| : x \in [0,1]\} = d_{\max}(f_n, f), \end{aligned}$$

for each $n \in \mathbb{N}$, so (M_n) is also a null sequence. Hence $f_n \to f$ uniformly as $n \to \infty$. ■

Thus, if we are given a sequence of functions (f_n) in $C[0,1]$, we can use our knowledge of the properties of uniform convergence to determine whether it is convergent for $d_{\max}$, as follows.

Strategy for determining convergence of functions in $C[0,1]$

1. Identify the possible limit function $f \in C[0,1]$ by finding the pointwise limit of (f_n). If there is no pointwise limit, then there is no uniform limit, and Theorem 3.3 tells us that (f_n) is not convergent with respect to $d_{\max}$.
2. If the pointwise limit of (f_n) exists, determine whether it is continuous. If it is not continuous, then Lemma 3.1 tells us that the only possible uniform limit is not continuous, and Theorems 3.2 and 3.3 together tell us that (f_n) is not convergent with respect to $d_{\max}$.
3. If f is the pointwise limit of (f_n) and is continuous, determine whether $(d_{\max}(f_n, f))$ is a null sequence. If it is, then by definition the sequence (f_n) converges to f for $d_{\max}$. Otherwise it does not converge in $C[0,1]$ for $d_{\max}$.

Worked problem 3.3

Consider the sequence of functions (f_n) where $f_n\colon [0,1] \to \mathbb{R}$ is given by

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

(a) For each $x \in [0,1]$, find $\lim_{n\to\infty} f_n(x)$.

(b) Use the above strategy to determine whether the sequence (f_n) converges in $C[0,1]$ for $d_{\max}$.

Solution

(a) Sketches of the graphs of f_n for the first few values of n (Figure 3.4) suggest that, for all $x > 0$, $f_n(x)$ is eventually 0, whereas if $x = 0$, then $f_n(x) = 1$ for all $n \in \mathbb{N}$.

Indeed, if $0 < x \le 1$, then $f_n(x) = 0$, for $n > 1/x$, and hence $f_n(x) \to 0$ as $n \to \infty$.

Also, $f_n(0) = 1$ for all $n \in \mathbb{N}$, and so $f_n(0) \to 1$ as $n \to \infty$.

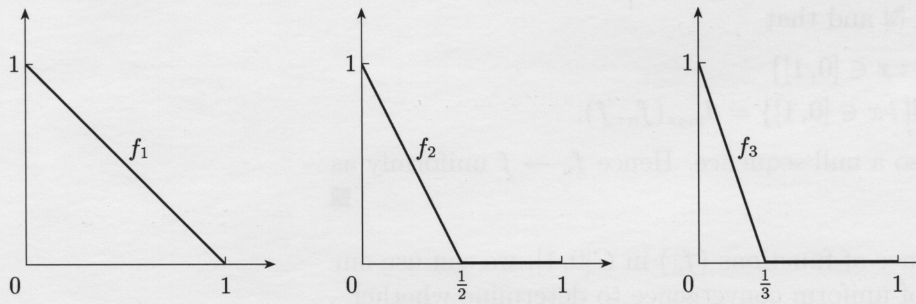


Figure 3.4

(b) We use the strategy given above. Step 1 requires us to identify a possible limit function. By (a), the only possible limit function for the sequence (f_n) is the pointwise limit function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \le 1. \end{cases}$$

But this function f is *not* continuous on $[0,1]$, and so is not in $C[0,1]$. Hence (f_n) does not converge in $C[0,1]$ for $d_{\max}$. ■

Problem 3.6

Consider the sequence of functions (f_n) where $f_n\colon [0,1] \to \mathbb{R}$ is given by

$$f_n(x) = \begin{cases} 2nx & \text{if } 0 \le x < \frac{1}{2n}, \\ 2 - 2nx & \text{if } \frac{1}{2n} \le x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

(a) Sketch the graphs of f_1, f_2 and f_3.

(b) For each $x \in [0,1]$, find $\lim_{n\to\infty} f_n(x)$.

(c) Determine whether the sequence (f_n) converges in $C[0,1]$ for $d_{\max}$.

4 Sequences and compact sets in metric spaces

After working through this section, you should be able to:

- define the terms *subsequence*, *convergent subsequence* and *sequentially compact*;
- appreciate that in a compact set any sequence has a convergent subsequence;
- appreciate that, in a metric space, compactness is equivalent to sequential compactness.

In this section we investigate the relationship between sequences and compact sets in metric spaces. Once we have understood this relationship, we can identify some compact infinite subsets of $C[0,1]$ for $d_{\max}$ — this is the theme of Section 5.

Recall from *Unit C2* that any *finite* set is compact.

4.1 Sequential compactness

The key to understanding the relationship between sequences and compact sets involves the notion of *convergent subsequences.*

First, we give the definition of a subsequence, which is the generalization of the definition of subsequences of real sequences to sequences in an arbitrary set.

Unit A1, Subsection 1.3.

Definition

Let (a_n) be a sequence in a set X. The sequence $(a_{n_k})_{k=1}^{\infty}$ is a **subsequence** of (a_n) if $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers — that is, if $1 \leq n_1 < n_2 < n_3 < \cdots$.

For example, $a_2, a_4, a_6, \ldots$ is a subsequence of $a_1, a_2, a_3, \ldots$.

Our first result is that any subsequence of a convergent sequence is also convergent, with the same limit. This is a generalization of Theorem 1.3 of *Unit A1.*

Theorem 4.1

Let $(X, \mathcal{T})$ be a topological space. If the sequence (a_n) converges to $a \in X$ for $\mathcal{T}$, then every subsequence (a_{n_k}) converges to a for $\mathcal{T}$.

Problem 4.1

Prove Theorem 4.1.

> ***Definition***
>
> Let $(X, \mathcal{T})$ be a topological space and let (a_n) be a sequence in X. The sequence (a_n) has a **convergent subsequence** in X if there is a subsequence of (a_n) that converges in X for $\mathcal{T}$.

In general, if $(X, \mathcal{T})$ is a topological space, then a sequence in the set X need not have any convergent subsequences. For example, if $a_n = n$ for all $n \in \mathbb{N}$, then (a_n) has no convergent subsequences for the Euclidean topology on $\mathbb{R}$. However, if (X, d) is a compact metric space, then every sequence in X has a convergent subsequence.

> ***Theorem 4.2***
>
> Let (X, d) be a compact metric space. Then each sequence in X has a convergent subsequence.

Proof The proof is by contradiction, twice.

Suppose that (X, d) is a compact metric space for which there is a sequence (a_n) in X with no convergent subsequences.

We claim that, under this supposition, for each $x \in X$, there are an $\varepsilon_x > 0$ and an $N_x \in \mathbb{N}$ such that

$$a_n \notin B_d(x, \varepsilon_x) \quad \text{for all } n > N_x.$$

We prove this claim by contradiction. Suppose that the claim is not true. Then there is an $x \in X$ such that, for each $\varepsilon > 0$ and each $N \in \mathbb{N}$, there is some $n > N$ for which

$$a_n \in B_d(x, \varepsilon).$$

In particular, for $\varepsilon_1 = \frac{1}{2}$, there is an integer n_1 such that $n_1 > 1$ and

$$a_{n_1} \in B_d(x, \varepsilon_1) = B_d(x, \tfrac{1}{2}).$$

Similarly, for $\varepsilon_2 = \frac{1}{4}$, there is an integer $n_2 > n_1$ such that

$$a_{n_2} \in B_d(x, \varepsilon_2) = B_d(x, \tfrac{1}{4}).$$

Proceeding in this way, we find, for $\varepsilon_k = 2^{-k}$, that there is an integer n_k such that $n_k > n_{k-1} > \cdots > n_1$ and

$$a_{n_k} \in B_d(x, \varepsilon_k) = B_d(x, 2^{-k}).$$

By Theorem 1.1, this gives a subsequence (a_{n_k}) that converges to x. But we are assuming that no such subsequence exists — a contradiction!

Thus, under the supposition that the sequence (a_n) has no convergent subsequences, we have shown that, for each $x \in X$, there are an $\varepsilon_x > 0$ and an $N_x \in \mathbb{N}$ with

$$B_d(x, \varepsilon_x) \cap \{a_n : n > N_x\} = \varnothing.$$

Clearly the collection

$$\{B_d(x, \varepsilon_x) : x \in X\}$$

is an open cover of X. Since X is compact, the definition of compactness tells us that there exist $x_1, x_2, \ldots, x_k \in X$ such that

This is where we use compactness.

$$X \subseteq \bigcup_{i=1}^{k} B_d(x_i, \varepsilon_{x_i}).$$

In particular, the set $\{a_n : n \in \mathbb{N}\}$ of the sequence is contained in this finite union of balls:

$$\{a_n : n \in \mathbb{N}\} \subseteq \bigcup_{i=1}^{k} B_d(x_i, \varepsilon_{x_i}).$$

By the definition of N_{x_i},

$$B_d(x_i, \varepsilon_{x_i}) \cap \{a_n : n > N_{x_i}\} = \varnothing.$$

Now let $N = \max\{N_{x_1}, N_{x_2}, \dots, N_{x_k}\}$. It follows that

$$a_n \notin \bigcup_{i=1}^{k} B_d(x_i, \varepsilon_{x_i}) \quad \text{for } n > N$$

— a contradiction, since the whole space is contained in this union!

We conclude that there is no sequence in X without a convergent subsequence. ■

Remark

We know that a compact subset of a metric space defines a compact metric subspace. Thus Theorem 4.2 can be extended to compact subsets of metric spaces:

if (X, d) is a metric space and (a_n) lies in a compact subset A of X, then (a_n) has a convergent subsequence with limit in A.

Problem 4.2

Let (a_n) be a bounded sequence in $\mathbb{R}$. Show that (a_n) has a convergent subsequence for the Euclidean metric on $\mathbb{R}$.

(a_n) is **bounded** if there is a $K \in \mathbb{R}$ with $|a_n| < K$ for all n.

The above remark motivates the following definition.

Definition

Let $(X, \mathcal{T})$ be a topological space. A set $A \subseteq X$ is **sequentially compact** if each sequence in A has a subsequence that converges to a point in A.

Remark

It is an essential part of this definition that the convergent subsequence converges to a point in the set A under consideration.

Theorem 4.2 shows that for metric spaces, each compact set is sequentially compact. In fact, in metric spaces, the two notions are equivalent: *a set is compact if and only if it is sequentially compact.* To demonstrate this, it remains to prove the following theorem.

Theorem 4.3

If (X, d) is a sequentially compact metric space, then (X, d) is compact.

The proof takes up the whole of Subsection 4.2. In Section 5, we see how useful this result is, by using it to identify some compact subsets of $C[0, 1]$ for $d_{\max}$.

4.2 Sequential compactness implies compactness

The main idea of the proof of Theorem 4.3 is to show that in a sequentially compact metric space it is possible, for each $\varepsilon > 0$, to find a finite set S of points such that each point in the space can be approximated to within a distance ε by some point of S. This allows us to 'replace' the whole space by the finite set of points, with an error of at most ε.

The proof of this result is quite long, and you may wish to omit this subsection on a first reading.

Definition

Let (X, d) be a metric space. For $\varepsilon > 0$, an **ε-net** for X is a set $S \subseteq X$ such that

$$X \subseteq \bigcup_{s \in S} B_d(s, \varepsilon).$$

Remarks

(i) An ε-net provides an approximation to within a distance ε of any point in the space. For example, $\mathbb{Z}$ is an ε-net for $\mathbb{R}$ with the Euclidean metric, provided that $\varepsilon > \frac{1}{2}$.

(ii) For each $\varepsilon > 0$, X is an ε-net for itself.

(iii) If S is an ε-net for X, then it is also a δ-net for X for each $\delta > \varepsilon$.

Lemma 4.4

Let (X, d) be a sequentially compact metric space. Then X has a *finite* ε-net for each $\varepsilon > 0$.

Proof Suppose that X is sequentially compact, but that there is an $\varepsilon > 0$ for which there is no finite ε-net for X. We obtain a contradiction by constructing a sequence in X that has no convergent subsequence.

Let $a_1 \in X$. Since there is no finite ε-net, $\{a_1\}$ cannot be an ε-net for X. Hence there is an $a_2 \in X$ such that $d(a_1, a_2) \geq \varepsilon$.

Similarly, $\{a_1, a_2\}$ cannot be an ε-net for X, and so there is an $a_3 \in X$ whose distance is at least ε from both a_1 and a_2 — that is,

$$\min\{d(a_1, a_3), d(a_2, a_3)\} \geq \varepsilon.$$

Thus for $\{a_1, a_2, a_3\}$, we have $d(a_i, a_j) \geq \varepsilon$ for each $i \neq j$.

We continue in this way, constructing subsets $\{a_1, a_2, \ldots, a_k\}$ of X that are not ε-nets for X and for which $d(a_i, a_j) \geq \varepsilon$ for $i \neq j$.

Thus we construct a sequence (a_n) such that $d(a_i, a_j) \geq \varepsilon$ for any two distinct terms a_i and a_j in the sequence. In particular, the terms of this sequence do not get arbitrarily close (they are always at least ε apart), and so this sequence has no convergent subsequences, contradicting our assumption that X is sequentially compact. ■

Problem 4.3

Use Lemma 4.4 to show that a sequentially compact metric space is bounded.

We defined boundedness for metric spaces in Subsection 3.2 of *Unit C2*.

You have just seen that if (X, d) is a metric space and there exists a finite ε-net for X, then (X, d) is bounded. If there exists a finite ε-net for X for *every* $\varepsilon > 0$, then we say that (X, d) is *totally bounded*.

Definition

A metric space (X, d) is **totally bounded** if there is a finite ε-net for each $\varepsilon > 0$.

Remark

Lemma 4.4 shows that each sequentially compact metric space is totally bounded.

Problem 4.4

Show that $[0, 1]$ (with the Euclidean metric) is totally bounded.

Problem 4.5

Show directly that each compact metric space (X, d) is totally bounded.

The next lemma is the key to the proof that sequential compactness implies compactness.

Lemma 4.5

Let (X, d) be a sequentially compact metric space and let $\mathcal{S}$ be an open cover of X. Then there is an $\varepsilon > 0$ such that, for each $x \in X$, there is a $U \in \mathcal{S}$ with $B_d(x, \varepsilon) \subseteq U$.

The number ε given in this lemma is often called a *Lebesgue number* of the open cover $\mathcal{S}$.

Henri Lebesgue (1875–1941) is primarily remembered for his fundamental contributions to the theory of integration.

Proof Let (X, d) be a sequentially compact metric space, and suppose that the result is false.

In this case, there is an open cover $\mathcal{S}$ of X such that, for each $\varepsilon > 0$, there is an $x \in X$ with

$$B_d(x, \varepsilon) - U \neq \varnothing \quad \text{for each } U \in \mathcal{S}.$$

This says that the ball $B_d(x, \varepsilon)$ is never entirely contained within an open set of the cover.

In particular, for each $n \in \mathbb{N}$, there is an $a_n \in X$ such that

$$B_d(a_n, \tfrac{1}{n}) - U \neq \varnothing \quad \text{for each } U \in \mathcal{S}.$$

Since X is sequentially compact, the sequence $(a_n)_{n=1}^{\infty}$ has a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ with limit $a \in X$, say.

Since $\mathcal{S}$ is a cover of X, there is a set $U \in \mathcal{S}$ with $a \in U$. Since U is open, there is an $M \in \mathbb{N}$ such that

$$B_d\left(a, \tfrac{2}{M}\right) \subseteq U.$$

Now, $a_{n_k} \to a$ as $k \to \infty$, and so there is a $K \in \mathbb{N}$ such that, for all $k > K$,

$$d(a_{n_k}, a) < \tfrac{1}{M}, \quad \text{and so } a_{n_k} \in B_d\left(a, \tfrac{1}{M}\right).$$

We now recall that, by the definition of the sequence (a_n),

$$B_d\left(a_{n_k}, \tfrac{1}{n_k}\right) - U \neq \varnothing \quad \text{for each } k \in \mathbb{N}.$$

However, if k is chosen so that both $k > K$ and $n_k > M$, then

$$B_d\left(a_{n_k}, \tfrac{1}{n_k}\right) \subseteq B_d\left(a_{n_k}, \tfrac{1}{M}\right) \subseteq B_d\left(a, \tfrac{2}{M}\right) \subseteq U.$$

See Figure 4.1.

This implies that $B_d\left(a_{n_k}, \tfrac{1}{n_k}\right) - U = \varnothing$ — a contradiction.

Hence there is an $\varepsilon > 0$ such that, for each $x \in X$, there is a $U \in \mathcal{S}$ with $B_d(x, \varepsilon) \subseteq U$.

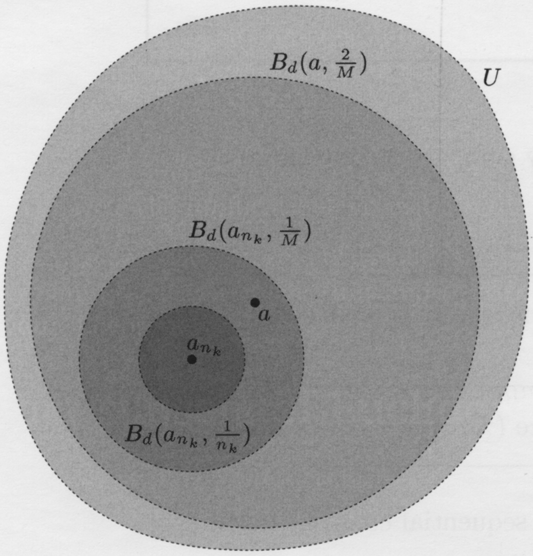


Figure 4.1 ■

We can now prove that sequentially compact metric spaces are compact.

Completing the proof of Theorem 4.3

Proof Let (X, d) be a sequentially compact metric space. We must show that each open cover of X has a finite subcover.

So let $\mathcal{S}$ be an open cover of X. Lemma 4.5 implies that there is an $\varepsilon > 0$ such that, for each $x \in X$, there is a $U \in \mathcal{S}$ with $B_d(x, \varepsilon) \subseteq U$.

Since (X, d) is sequentially compact, Lemma 4.4 implies that (X, d) is totally bounded. Thus there are a finite number of points $x_1, x_2, \ldots, x_N \in X$ such that, for this ε,

$$X \subseteq \bigcup_{i=1}^{N} B_d(x_i, \varepsilon).$$

But, for each x_i, there is a $U_i \in \mathcal{S}$ for which $B_d(x_i, \varepsilon) \subseteq U_i$. Therefore

$$X \subseteq \bigcup_{i=1}^{N} B_d(x_i, \varepsilon) \subseteq \bigcup_{i=1}^{N} U_i.$$

Thus the collection $\{U_1, U_2, \ldots, U_N\}$ is a finite subcover of X from $\mathcal{S}$.

Since $\mathcal{S}$ is an arbitrary open cover of X, the result follows. ■

Thus, for a metric space, sequential compactness and compactness are equivalent notions:

a metric space is compact if and only if it is sequentially compact.

5 Compact subsets of $C[0,1]$

After working through this section, you should be able to:
- ► identify some compact subsets of $C[0,1]$ for $d_{\max}$;
- ► solve some extreme-value problems on $C[0,1]$ for $d_{\max}$.

We already know some examples of compact subsets of $C[0,1]$ for $d_{\max}$. For example, any finite set is compact. In this section, we use our characterization of compact sets in terms of convergent subsequences to identify further examples, and we develop a criterion that makes it relatively straightforward to test whether a given subset of $C[0,1]$ is compact.

5.1 Identifying compact subsets of $C[0,1]$

In the previous section, we showed that a subset of a metric space is compact if and only if it is sequentially compact. This gives us a method for showing that a given set is *not* compact:

> if there is a sequence in the set that has no subsequences converging to a point in the set, then the set is not compact.

Worked problem 5.1

Let

$$A = \{f \in C[0,1] : |f(x)| \leq 1 \text{ for all } x \in [0,1]\}.$$

Show that A is not compact for $d_{\max}$.

Solution

It is sufficient to find a sequence of functions in A that has no subsequences converging to a point in A.

With this in mind, we consider the sequence of functions (f_n) in A given by

You met this sequence of functions in Worked problem 3.3.

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

The first few functions in the sequence are shown in Figure 5.1 overleaf.
We know that, for $x \in [0,1]$,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

We calculated the limit in Worked problem 3.3.

Thus the pointwise limit of the sequence is a discontinuous function. Moreover, any subsequence of (f_n) has the same pointwise limit. Hence no subsequence can converge to a continuous function. We deduce from Theorem 4.2 that A is not compact.

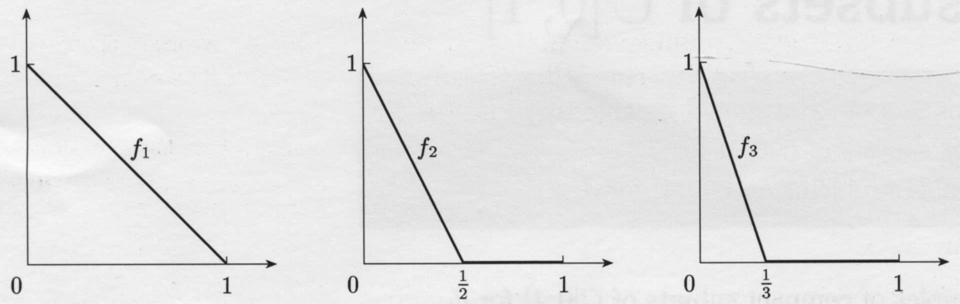


Figure 5.1

■

Notice that the set A in Worked problem 5.1 is both closed and bounded for $d_{\max}$. So, for $d_{\max}$, the compact subsets of $C[0,1]$ *do not* coincide with the closed and bounded subsets.

This contrasts with the situation in $(\mathbb{R}^n, d^{(n)})$ (see *Unit C2*, Theorem 4.2).

Problem 5.1

Let

$$A = \{f \in C[0,1] : |f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in [0,1]\}.$$

Show that A is not compact for $d_{\max}$.

Hint Find a sequence (f_n) of functions in A such that (f_n) and any subsequence of (f_n) has no pointwise limit, and then make use of results from Section 3.

We now state a result that gives us sufficient conditions to guarantee that a subset of $C[0,1]$ *is* compact.

> **Theorem 5.1**
>
> Let A be a closed subset of $C[0,1]$ for $d_{\max}$. Suppose that:
>
> (a) for each $x \in [0,1]$, there is an $M_x \geq 0$ such that $|f(x)| \leq M_x$, for each $f \in A$;
>
> (b) for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for each $f \in A$ and each $a \in [0,1]$,
>
> $$|f(x) - f(a)| < \varepsilon \quad \text{whenever } x \in [0,1] \text{ and } |x - a| < \delta.$$
>
> Then A is compact.

A is a *pointwise bounded* family of functions.

A is an *equicontinuous* family of functions — the value of δ in the definition of continuity does not depend on the point a or the function f.

We defer the proof to Subsection 5.2. Here we see how we can make use of the result.

Worked problem 5.2

Let

$$A = \{f \in C[0,1] : f(0) = 0 \text{ and } |f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in [0,1]\}.$$

Define $I\colon C[0,1] \to \mathbb{R}$ by

$$I(f) = \int_0^1 f(x)\,dx.$$

Show that I attains a minimum on A.

Solution

In *Unit A2* we proved that I is a $(d_{\max}, d^{(1)})$-continuous function. If we can prove that A is a compact set, the General Extreme Value Theorem then implies that I is bounded on A and attains its bounds. In particular, there is an $f_0 \in A$ such that

Unit A2, Worked problem 2.4.
Unit C2, Theorem 4.3.

$$I(f_0) \leq I(f) \quad \text{for all } f \in A,$$

In fact, f_0 is the function given by $f_0(x) = -x$, but we are not asked to show this.

and so I attains its minimum on A.

We have thus achieved our aim if we can show that A is compact. To do this, we use Theorem 5.1 and verify that:

(i) A is closed;
(ii) A is a pointwise bounded family of functions;
(iii) A is an equicontinuous family of functions.

(i) A *is closed.*

Let (f_n) be a sequence of functions in A that converges to $f \in C[0,1]$ for $d_{\max}$. By Theorem 2.2, in order to show that A is closed, we must show that $f \in A$.

Since $f_n(0) = 0$ for all $n \in \mathbb{N}$,

$$|f(0)| = |f(0) - f_n(0)| \leq d_{\max}(f_n, f).$$

But $d_{\max}(f_n, f) \to 0$ as $n \to \infty$. Hence $f(0) = 0$.

For $x, y \in [0,1]$, the Triangle Inequality implies that, for each $n \in \mathbb{N}$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq |f_n(x) - f_n(y)| + 2d_{\max}(f_n, f) \\ &\leq |x - y| + 2d_{\max}(f_n, f), \end{aligned}$$

since $f_n \in A$. But $n \in \mathbb{N}$ is arbitrary and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } x, y \in [0,1].$$

Hence $f \in A$ and so A is closed.

(ii) A *is a pointwise bounded family of functions.*

Let $x \in [0,1]$. Then, for each $f \in A$,

$$|f(x)| = |f(x) - f(0)| \leq |x - 0| = |x| \leq 1.$$

This bound does not depend on the choice of f, and so A is a pointwise bounded family of functions.

In fact, this bound does not depend on the choice of x either, but we need only independence of the choice of f for pointwise boundedness.

(iii) A *is an equicontinuous family of functions.*

Let $\varepsilon > 0$ and fix $a \in [0,1]$.
If $x \in [0,1]$ and $|x - a| < \delta$ then, for each $f \in A$,

$$|f(x) - f(a)| \leq |x - a| < \delta.$$

Take $\delta = \varepsilon$. Then, for each $f \in A$ and each $a \in [0,1]$, $|f(x) - f(a)| < \varepsilon$ whenever $x \in [0,1]$ satisfies $|x - a| < \delta$. Thus A is an equicontinuous family of functions.

Since the hypotheses of Theorem 5.1 hold, A is compact. ■

Problem 5.2

Let

$$A = \{f \in C[0,1] : f(0) = 0 \text{ and } |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for all } x, y \in [0,1]\}.$$

Define $F\colon C[0,1] \to \mathbb{R}$ by

$$F(f) = \int_0^1 f(x)^2\, dx.$$

(a) Prove that A is compact.

(b) Prove that F is $(d_{\max}, d^{(1)})$-continuous, and deduce that F attains a minimum on A.

5.2 Proof of Theorem 5.1

In this subsection, we prove Theorem 5.1.

This subsection is not assessed.

Suppose that A is a closed subset of $C[0,1]$ for $d_{\max}$ and that:

(a) for each $x \in [0,1]$, there is an $M_x \geq 0$ such that $|f(x)| \leq M_x$, for each $f \in A$;

(b) for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for each $f \in A$ and each $a \in [0,1]$, $|f(x) - f(a)| < \varepsilon$ whenever $x \in [0,1]$ and $|x - a| < \delta$.

We show that A is sequentially compact — that is, each sequence of functions in A has a $d_{\max}$-convergent subsequence with limit in A. It then follows from Theorem 4.3 that A is compact.

So let (f_n) be a sequence of functions in A — we aim to find a $d_{\max}$-convergent subsequence. Our objective is to 'build' a continuous function f to which a subsequence of (f_n) converges. We do this in two stages:

1 we find a candidate function f to which a subsequence of (f_n) may converge;

2 we show that a subsequence of (f_n) converges for $d_{\max}$ to this function f, and at the same time show that $f \in A$.

Finding a candidate limit function f

For each $a \in [0,1]$, there is a sequence $(n_k)_{k\in\mathbb{N}}$ with $1 \leq n_1 < n_2 < n_3 < \cdots$ such that $(f_{n_k}(a))$ is a convergent sequence of real numbers. This follows from assumption (a), which tells us that $(f_n(a))$ is a bounded sequence of real numbers, and so lies in a compact interval $[-M_a, M_a]$ for some $M_a \geq 0$. Theorem 4.2 then implies that $(f_n(a))$ has a convergent subsequence.

Now let (a_i) be a sequence of real numbers whose set is dense in $[0,1]$ — that is,

Dense sets were defined in *Unit A4*, Subsection 3.1.

$$\mathrm{Cl}(\{a_i : i \in \mathbb{N}\}) = [0,1];$$

an example of such a sequence is

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \ldots, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, \tfrac{n-1}{n}, \tfrac{1}{n+1}, \ldots.$$

This sequence contains every rational in $(0,1)$.

We aim to find a strictly increasing sequence (n_k) such that $(f_{n_k}(a_i))$ is a convergent sequence for each $i \in \mathbb{N}$. We do this by induction.

We start by choosing a strictly increasing sequence $(n_k^1)_{k\in\mathbb{N}}$ for which $(f_{n_k^1}(a_1))_{k\in\mathbb{N}}$ is a convergent sequence.

We have shown above that we can do this.

Now suppose that, for some i, we have a strictly increasing sequence $(n_k^i)_{k\in\mathbb{N}}$, such that $(f_{n_k^i}(a_j))_{k\in\mathbb{N}}$ is a convergent sequence of real numbers

for $j = 1, 2, \ldots, i$. Then again we can find a subsequence $(n_k^{i+1})_{k\in\mathbb{N}}$ of $(n_k^i)_{k\in\mathbb{N}}$ such that $(f_{n_k^{i+1}}(a_{i+1}))_{k\in\mathbb{N}}$ is also convergent.

This subsequence is also strictly increasing.

Proceeding by induction, we find a sequence of strictly increasing sequences $((n_k^i)_{k\in\mathbb{N}})_{i\in\mathbb{N}}$ for which:

(i) $(n_k^{i+1})_{k\in\mathbb{N}}$ is a subsequence of $(n_k^i)_{k\in\mathbb{N}}$, for each $i \in \mathbb{N}$;

(ii) $n_k^k < n_{k+1}^{k+1}$ for each $k \in \mathbb{N}$, since $n_k^k \le n_k^{k+1} < n_{k+1}^{k+1}$;

(iii) for each $i \in \mathbb{N}$, $(f_{n_k^i}(a_j))_{k\in\mathbb{N}}$ is convergent for $j = 1, 2, \ldots, i$.

We now consider the diagonal sequence $(N_k)_{k=1}^{\infty}$, where $N_k = n_k^k$. Then, for each $i \in \mathbb{N}$, $(f_{N_k}(a_i))_{k\in\mathbb{N}}$ is a convergent sequence, since $(N_k)_{k=i}^{\infty}$ is a subsequence of $(n_k^i)_{k\in\mathbb{N}}$.

In *Unit A3*, we used a diagonalization argument to show that $(0, 1)$ is uncountable.

We can now begin to define the function f that will turn out to be the $d_{\max}$-limit of the sequence (f_{N_k}). We start by defining it for each a_i in our sequence (a_i), by setting

$$f(a_i) = \lim_{k\to\infty} f_{N_k}(a_i).$$

This gives a function $f\colon \{a_i : i \in \mathbb{N}\} \to \mathbb{R}$.

We now extend the definition of f to arbitrary points in $[0, 1]$. This is where we use the fact that $\{a_i : i \in \mathbb{N}\}$ is dense in $[0, 1]$. By Theorem 2.2, given $a \in [0, 1]$, there is a subsequence (a_{n_i}) of (a_i) for which $a_{n_i} \to a$. We wish to use this subsequence to define $f(a)$ by setting

$$f(a) = \lim_{i\to\infty} f(a_{n_i}).$$

Unfortunately there are two potential problems:

(i) given a subsequence (a_{n_i}) of (a_i) that converges to a, it is not clear that $(f(a_{n_i}))$ is a convergent sequence;

(ii) even if $(f(a_{n_i}))$ is convergent, there may be another subsequence (a_{m_i}) that converges to a with $(f(a_{m_i}))$ having a different limit — this would give us a different choice for defining $f(a)$.

We use assumption (b) to show that neither problem can arise.

We first show that if (a_{n_i}) and (a_{m_i}) are subsequences of (a_n) that converge to a as $i \to \infty$, then $|f(a_{n_i}) - f(a_{m_i})| \to 0$ as $i \to \infty$.

To see this, let $\varepsilon > 0$ be given, and choose a $\delta > 0$ so that, for each $g \in A$ and each $a \in [0, 1]$,

We are using (b) here.

$$|g(x) - g(a)| < \varepsilon/4 \quad \text{whenever } x \in [0, 1] \text{ satisfies } |x - a| < \delta. \qquad (*)$$

The Triangle Inequality implies that, for any $N \in \mathbb{N}$,

$$\begin{aligned}|f(a_{n_i}) - f(a_{m_i})| &\le |f(a_{n_i}) - f_N(a_{n_i})| + |f_N(a_{n_i}) - f_N(a)| \\ &\quad + |f_N(a) - f_N(a_{m_i})| + |f_N(a_{m_i}) - f(a_{m_i})|.\end{aligned}$$

Since $a_{n_i}, a_{m_i} \to a$ as $i \to \infty$, there is an $M \in \mathbb{N}$ such that, for $i > M$, both $|a_{n_i} - a| < \delta$ and $|a_{m_i} - a| < \delta$. For $i > M$, we then have by $(*)$, for *any* $N \in \mathbb{N}$,

$$|f_N(a_{n_i}) - f_N(a)| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad |f_N(a) - f_N(a_{m_i})| < \tfrac{1}{4}\varepsilon.$$

Since, for each $i \in \mathbb{N}$, $f_N(a_{n_i}) \to f(a_{n_i})$ and $f_N(a_{m_i}) \to f(a_{m_i})$ as $N \to \infty$, for each $i > M$ there is an $N \in \mathbb{N}$ for which

$$|f(a_{n_i}) - f_N(a_{n_i})| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad |f_N(a_{m_i}) - f(a_{m_i})| < \tfrac{1}{4}\varepsilon.$$

Hence, fixing some $i > M$ and choosing an $N \in \mathbb{N}$ so that this estimate holds, we deduce that

$$|f(a_{n_i}) - f(a_{m_i})| \le \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$|f(a_{n_i}) - f(a_{m_i})| \to 0 \text{ as } i \to \infty. \tag{5.1}$$

Thus, if (a_{n_i}) and (a_{m_i}) converge to a and $(f(a_{n_i}))$ converges, then $f((a_{m_i}))$ converges to the same value.

We now show that if $a_{n_i} \to a$ as $n \to \infty$, then $\lim_{i\to\infty} f(a_{n_i})$ exists.

Suppose that $a_{n_i} \to a$ as $n \to \infty$. Then, by the Triangle Inequality,

$$|f(a_{n_i})| \le |f(a_{n_i}) - f_N(a_{n_i})| + |f_N(a_{n_i}) - f_N(a)| + |f_N(a)|,$$

for each $N \in \mathbb{N}$. However, $(f_N(a))$ is bounded, and so there is an $M_a > 0$ such that $|f_N(a)| \le M_a$ for each $N \in \mathbb{N}$. Moreover, there is a $\delta > 0$ such that, if $x \in [0,1]$ satisfies $|x - a| < \delta$, then $|f_N(x) - f_N(a)| < 1$ for each $N \in \mathbb{N}$. So, on choosing $I \in \mathbb{N}$ so that $|a_{n_i} - a| < \delta$ for $i > I$, we find that, for each $i > I$, there is an N_k so that $|f(a_{n_i}) - f_{N_k}(a_{n_i})| < 1$, and so, for such an $i > I$,

Here we are using (a).

Here we are using (b).

$$|f(a_{n_i})| \le 1 + 1 + M_a = 2 + M_a.$$

Hence, for each $i \in \mathbb{N}$,

$$|f(a_{n_i})| \le \max\{|f(a_{n_i})| : i = 1, 2, \ldots, I\} + 2 + M_a,$$

and so $(f(a_{n_i}))$ is a bounded sequence of real numbers. Thus, as before, we can find a convergent subsequence $(f(a_{n_{i(j)}}))_{j=1}^{\infty}$ with limit $f(a)$ (say) as $j \to \infty$. It now follows from (5.1) that

$$|f(a_{n_i}) - f(a)| \le |f(a_{n_i}) - f(a_{n_{i(j)}})| + |f(a_{n_{i(j)}}) - f(a)| \to 0 \text{ as } i, j \to \infty,$$

and so $\lim_{i\to\infty} f(a_{n_i})$ exists and is $f(a)$. This completely defines f.

A subsequence of (f_n) converges to f

It remains to show that $f \in A$. If we can show that $f_{N_k} \to f$ uniformly as $k \to \infty$, then Theorem 3.2 and Theorem 3.3 imply that $f \in C[0,1]$, with $f_{N_k} \to f$ for $d_{\max}$. This gives $f \in A$, since A is closed (Theorem 2.2).

We split the proof that $f_{N_k} \to f$ uniformly into three steps:

(1) $f_{N_k}(a) \to f(a)$ for each $a \in [0,1]$;

(2) for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(b) - f(a)| < \varepsilon$ whenever $a, b \in [0,1]$ and $|b - a| < \delta$;

(3) steps (1) and (2) imply that $f_{N_k} \to f$ uniformly as $k \to \infty$.

Step (1): *$f_{N_k}(a) \to f(a)$ for each $a \in [0,1]$.*

Let $a \in [0,1]$, and let $\varepsilon > 0$ be given. We must show that there is a $K \in \mathbb{N}$ such that $|f(a) - f_{N_k}(a)| < \varepsilon$ for all $k > K$.

First, choose $\delta > 0$ so that, whenever $x \in [0,1]$ with $|x - a| < \delta$, then $|f_N(x) - f_N(a)| < \frac{1}{3}\varepsilon$ for any $N \in \mathbb{N}$.

Here we are using (b).

Since (a_i) is dense in $[0,1]$, there is a subsequence (a_{n_i}) for which $a_{n_i} \to a$ as $i \to \infty$. And so by (5.1) $f(a_{n_i}) \to f(a)$ as $i \to \infty$. Hence there is an $I \in \mathbb{N}$ such that $|a_{n_i} - a| < \delta$ and $|f(a_{n_i}) - f(a)| < \frac{1}{3}\varepsilon$ for $i > I$.

Hence, for such an $i > I$, the Triangle Inequality gives

$$\begin{aligned}|f(a) - f_{N_k}(a)| &\le |f(a) - f(a_{n_i})| + |f(a_{n_i}) - f_{N_k}(a_{n_i})| + |f_{N_k}(a_{n_i}) - f_{N_k}(a)| \\ &< \tfrac{1}{3}\varepsilon + |f(a_{n_i}) - f_{N_k}(a_{n_i})| + \tfrac{1}{3}\varepsilon.\end{aligned}$$

But $f_{N_k}(a_{n_i}) \to f(a_{n_i})$ as $k \to \infty$, and so there is a $K \in \mathbb{N}$ so that $|f(a_{n_i}) - f_{N_k}(a_{n_i})| < \frac{1}{3}\varepsilon$ for all $k > K$. Hence

$$|f(a) - f_{N_k}(a)| < \varepsilon \quad \text{for all } k > K,$$

and so $f_{N_k}(a) \to f(a)$ as $k \to \infty$.

Step (2): *For each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that* $|f(b) - f(a)| < \varepsilon$ *whenever* $a, b \in [0,1]$ *and* $|b - a| < \delta$.

Let $\varepsilon > 0$ be given and choose a $\delta > 0$ so that, for each $a \in [0,1]$ and $b \in [0,1]$ with $|b-a| < \delta$, $|f_N(b) - f_N(a)| < \frac{1}{3}\varepsilon$ for all $N \in \mathbb{N}$. For such points a and b, the Triangle Inequality gives

Here we are using (b).

$$\begin{aligned}|f(b) - f(a)| &\le |f(b) - f_{N_k}(b)| + |f_{N_k}(b) - f_{N_k}(a)| + |f_{N_k}(a) - f(a)| \\ &< |f(b) - f_{N_k}(b)| + \tfrac{1}{3}\varepsilon + |f_{N_k}(a) - f(a)|.\end{aligned}$$

But, by (1), $f_{N_k}(b) \to f(b)$ and $f_{N_k}(a) \to f(a)$. Hence there is a $K \in \mathbb{N}$ so that, for all $k > K$, both $|f(b) - f_{N_k}(b)| < \frac{1}{3}\varepsilon$ and $|f_{N_k}(a) - f(a)| < \frac{1}{3}\varepsilon$. Hence, for all $k > K$,

$$|f(b) - f(a)| < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.$$

Step (3): $f_{N_k} \to f$ *uniformly as* $k \to \infty$.

Let $\varepsilon > 0$ be given. We must show that there is a $K \in \mathbb{N}$ so that $|f(a) - f_{N_k}(a)| < \varepsilon$, for all $k > K$ and all $a \in [0,1]$.

To do this, choose a $\delta > 0$ so that, for all $N \in \mathbb{N}$, $|f_N(b) - f_N(a)| < \frac{1}{3}\varepsilon$ and $|f(b) - f(a)| < \frac{1}{3}\varepsilon$ whenever $a, b \in [0,1]$ with $|b - a| < \delta$. Let $B = \{j\delta : j = 0, 1, 2, \ldots\} \cap [0,1]$. Then B is a finite set and, for each $a \in [0,1]$, there is a $b \in B$ with $|b - a| < \delta$. For each $b \in B$, by (1) there is $K_b \in \mathbb{N}$ so that $|f(b) - f_{N_k}(b)| < \frac{1}{3}\varepsilon$ for all $k > K_b$. Let $K = \max\{K_b : b \in B\}$. Now let $a \in [0,1]$ and choose a $b \in B$ so that $|b - a| < \delta$. By the Triangle Inequality, for all $k > K$,

Here we are using (b) and Step 2.

K is finite, since B is finite.

$$\begin{aligned}|f(a) - f_{N_k}(a)| &\le |f(a) - f(b)| + |f(b) - f_{N_k}(b)| + |f_{N_k}(b) - f_{N_k}(a)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.\end{aligned}$$

Hence $f_{N_k} \to f$ uniformly as $k \to \infty$, and Theorem 5.1 follows.

Thus we have shown that each sequence of functions in A has a convergent subsequence. We deduce from Theorem 4.3 that A is compact.

Solutions to problems

1.1 **(a)** When $X = \mathbf{C}$, the elements of X are infinite sequences of 0s and 1s, and so a sequence in X consists of an infinite sequence of infinite sequences! An example of such a sequence (a_n) is given by

$$a_n = (\underbrace{1, 1, 1, \ldots, 1}_{n \text{ 1s}}, 0, 0, \ldots).$$

(b) An example is $(f_1, f_2, f_3, \ldots)$, where, for $n \in \mathbb{N}$, f_n is a continuous function on $[0, 1]$ — for example, $f_n\colon [0, 1] \to \mathbb{R}$ given by $f_n(x) = nx$.

1.2 Let $\varepsilon > 0$. We begin by estimating $d^{(2)}((1, 2), a_n)$ for $n \in \mathbb{N}$. We have

$$\begin{aligned} d^{(2)}((1,2), (1 + \tfrac{1}{n}, 2 - \tfrac{1}{n^2})) &= \sqrt{(\tfrac{1}{n})^2 + (-\tfrac{1}{n^2})^2} \\ &\leq \sqrt{(\tfrac{1}{n})^2 + (\tfrac{1}{n})^2} = \tfrac{\sqrt{2}}{n}. \end{aligned}$$

So, if $N > \sqrt{2}/\varepsilon$, then for $n > N$,

$$d^{(2)}((1,2), (1 + \tfrac{1}{n}, 2 - \tfrac{1}{n^2})) \leq \tfrac{\sqrt{2}}{n} < \varepsilon$$

— that is, $a_n \in B_{d^{(2)}}((1, 2), \varepsilon)$ whenever $n > N$.

1.3 Let (a_n) converge (for the co-countable topology) to $a \in X$. Then (a_n) eventually lies in U for each neighbourhood U of a. A neighbourhood of a is a set containing a that has a countable complement.

In particular, since $\{a_n : n \in \mathbb{N}\}$ is countable,

$$U = \{a_n : n \in \mathbb{N}\}^c \cup \{a\}$$

is a neighbourhood of a. Since (a_n) converges to a, there is an $N \in \mathbb{N}$ such that

$$a_m \in \{a_n : n \in \mathbb{N}\}^c \cup \{a\},$$

for all $m > N$. This is possible only if $a_m = a$ for all $m > N$.

1.4 **(a)** Let (a_n) be a sequence in X. The only neighbourhood of a is X, and X contains every term of the sequence, so (a_n) converges to a.

(b) The sequence $(p, p, p, \ldots)$ converges to p, since every neighbourhood of p contains all points of the sequence. It also converges to a by part (a).

If $b \in X$ is distinct from both a and p, then $\{b\}$ is a neighbourhood of b that contains *no* points of the sequence, so the sequence does not converge to b.

1.5 Let (a_n) be a sequence in X, and suppose it converges to $a \in X$ with respect to $\mathcal{T}_2$. Let $f\colon X \to X$ be the identity map given by $f(x) = x$. Then by Theorem 6.1 of *Unit A3*, f is $(\mathcal{T}_2, \mathcal{T}_1)$-continuous, and Theorem 1.3 implies that $a_n \xrightarrow{\mathcal{T}_1} a$ as $n \to \infty$.

1.6 Let $X = \{a, b\}$ (with $a \neq b$), let $\mathcal{T}_1$ be the indiscrete topology on X and let $\mathcal{T}_2$ be the discrete topology on X. Then $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Let (a_n) be the sequence $(a, b, a, b, \ldots)$. Then, for $\mathcal{T}_1$, the sequence (a_n) converges to both a and b, whereas, for $\mathcal{T}_2$, it does not converge since it is not eventually constant.

1.7 **(a)** The component sequences $(1 - \frac{1}{n})$ and $(1 + \frac{1}{n})$ both converge to 1 for the Euclidean topology. Thus this sequence converges to $(1, 1)$ for the product topology on $\mathbb{R}^2$.

(b) The component sequence $(\cos n\pi) = (-1, 1, -1, 1, \ldots)$ does not converge, so this sequence does not converge.

(c) All three component sequences, $(3^n/n!)$, $((-1)^n/n)$ and $((n-5)/(n^2+2n+1))$, converge to 0 for the Euclidean topology on $\mathbb{R}$. Thus this sequence converges to $(0, 0, 0)$ for the product topology.

1.8 The first component sequence converges for the Euclidean topology, since it is a basic null sequence. However, the sequence $(\frac{1}{n^2})$ is not eventually constant and so, by the result of Worked problem 1.2, does not converge for the discrete topology. Hence $((1/n, 1/n^2))$ is divergent for this topology.

1.9 Let $(X, \mathcal{T})$ be a Hausdorff space, and suppose that $a, b \in X$. Let (a_n) be a sequence in X that converges to both a and b. Suppose that $a \neq b$.

Since $(X, \mathcal{T})$ is a Hausdorff space, there are open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \varnothing$.

Since $a_n \to a$, there is an $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

Similarly, since $a_n \to b$ there is an $M \in \mathbb{N}$ such that $a_n \in V$ for all $n > M$.

But then, for all $n > \max\{N, M\}$, $a_n \in U$ and $a_n \in V$; that is, $a_n \in U \cap V = \varnothing$, by our choice of U and V. This is a contradiction, and so $a = b$.

2.1 Since $\mathbf{a}_n$ and $\mathbf{a}$ first differ at the $(n+1)$th term,

$$d_{\mathbf{C}}(\mathbf{a}_n, \mathbf{a}) = \frac{1}{2^{n+1}}.$$

Since $(1/2^{n+1})$ is a null sequence, $\mathbf{a}_n \to \mathbf{a}$ as $n \to \infty$.

2.2 Let $\varepsilon > 0$. Since $a_n \xrightarrow{d} a$, there is an $N \in \mathbb{N}$ such that, for all $n > N$, $d(a_n, a) < \varepsilon$. But for $n > N$

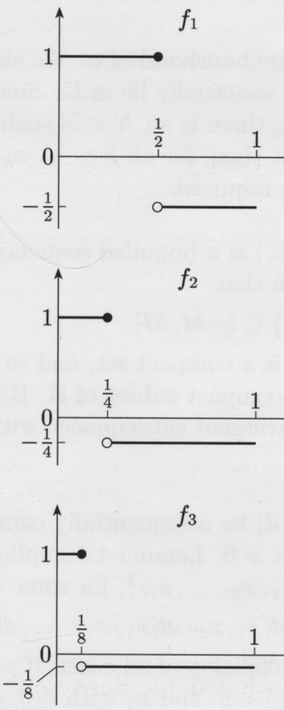
$$e(a_n, a) \leq d(a_n, a) < \varepsilon.$$

Thus $a_n \xrightarrow{e} a$.

2.3 **(a)** Since $A \subseteq \mathrm{Cl}(A)$, it follows that $\mathrm{Cl}(A)$ is uncountable. But $\mathrm{Cl}(A)$ is a closed set, and so is the complement of an open set of $\mathcal{T}$. However, the only open set in $\mathcal{T}$ with an uncountable complement is $\varnothing$, and so $\mathrm{Cl}(A) = [0, 1]$.

(b) Let $a \in (\frac{1}{2}, 1]$, and suppose that (a_n) is a sequence in A. Then $a_n \neq a$ for all n, and so $U = [0, 1] - \{a_n : n \in \mathbb{N}\}$ is a neighbourhood of a. But $\{a_n : n \in \mathbb{N}\} \cap U = \varnothing$, and so (a_n) does not eventually lie in U. Hence $a_n \not\to a$ for $\mathcal{T}$ as $n \to \infty$.

3.1 Sketches of the graphs of f_n for several values of n suggest that, for all $x > 0$, $f_n(x) \to 0$ as $n \to \infty$.



Indeed, for $x > 0$, choose N so that $\frac{1}{2^N} < x$; then, for all $n > N$, $f_n(x) = -1/2^n$. So, for $x > 0$, $f_n(x) \to 0$ as $n \to \infty$.

The behaviour at the point $x = 0$ is different: since $f_n(0) = 1$ for all $n \in \mathbb{N}$, $f_n(0) \to 1$ as $n \to \infty$.

So the pointwise limit of the sequence (f_n) is the function $f : [0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x < \infty. \end{cases}$$

3.2 **(a)** The graphs of f_1, f_2 and f_3 are shown in the right-hand column.

(b) We have $f_n(0) = 0$, for all $n \in \mathbb{N}$, and so $f_n(0) \to 0$ as $n \to \infty$.

If $x > 0$, then choose $N \in \mathbb{N}$ so that $1/N < x$. Then, for $n > N$, $f_n(x) = 0$. So $f_n(x) \to 0$ as $n \to \infty$.

Hence the pointwise limit function f of (f_n) is the zero function on $[0, 1]$. Thus $\int_0^1 f(x)\,dx = 0$.

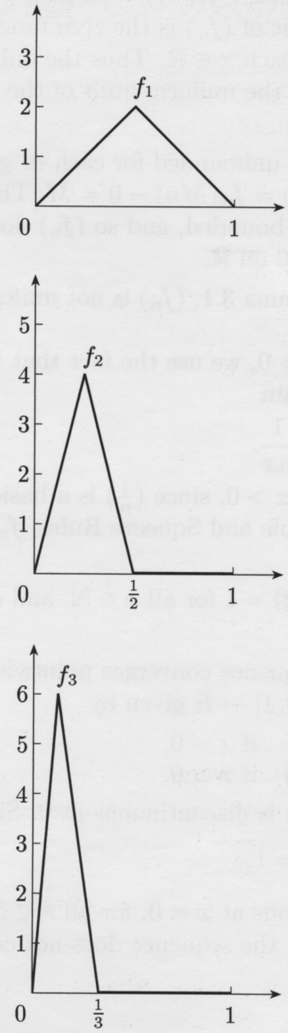
(c) For each $n \in \mathbb{N}$, we find the integral of f_n by noting that the graph of f_n is a triangle with height $2n$ and width $1/n$. Thus

$$\int_0^1 f_n(x)\,dx = 1,$$

and so

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1.$$

(d) From part (b), $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0 \neq 1$. We conclude that the processes of integration and taking pointwise limits cannot, in general, be interchanged.



3.3 We begin by identifying the pointwise limit of the sequence. For each $x \in [0, 1]$, $1 - (x/n)^2 \to 1$ as $n \to \infty$, and so the pointwise limit of the sequence (f_n) is the constant function $f: [0, 1] \to \mathbb{R}$ given by $f(x) = 1$ for each $x \in [0, 1]$.

Thus, by Lemma 3.1, our only candidate for the uniform limit of the sequence (f_n) is the function f. To show that it is the uniform limit, we show that there is an $N \in \mathbb{N} \cup \{0\}$ such that:

(i) the function $f_n - f$ is bounded for each $n > N$;

(ii) the sequence (M_n) defined by

$$M_n = \sup\{|f_{n+N}(x) - f(x)| : x \in [0, 1]\}$$

is a null sequence.

To prove (i), we observe that, for $x \in [0, 1]$ and each $n \in \mathbb{N}$,

$$0 \le |f_n(x) - f(x)| = |1 - (x/n)^2 - 1| = |x/n|^2 \le 1/n^2.$$

Hence $f_n - f$ is bounded for each $n > 0$.

For (ii), we note that, for each $n \in \mathbb{N}$,

$$0 \le M_n = \sup\{|1 - (x/n)^2 - 1| : x \in [0, 1]\} = 1/n^2.$$

Since $(1/n^2)$ is a basic null sequence, it follows that (M_n) is a null sequence.

Thus $f_n \to f$ uniformly as $n \to \infty$.

3.4 For each $x \in \mathbb{R}$, $x/n \to 0$ as $n \to \infty$, and so the pointwise limit of (f_n) is the zero function given by $f(x) = 0$ for each $x \in \mathbb{R}$. Thus the only possible candidate for the uniform limit of the sequence (f_n) is f.

But $f_n - f$ is unbounded for each n: given any $M > 0$, $(f_n - f)(Mn) = f_n(Mn) - 0 = M$. Thus, for each n, $f_n - f$ is not bounded, and so (f_n) does not converge uniformly to 0 on $\mathbb{R}$.

Hence by Lemma 3.1, (f_n) is not uniformly convergent.

3.5 For $x > 0$, we use the fact that $|\sin y| \leq 1$ for all $y \in \mathbb{R}$ to obtain

$$|f_n(x)| \leq \frac{1}{nx}.$$

Thus, for all $x > 0$, since $(\frac{1}{n})$ is a basic null sequence, by the Multiple and Squeeze Rules, $f_n(x) \to 0$ as $n \to \infty$.

We have $f_n(0) = 1$ for all $n \in \mathbb{N}$, and so $f_n(0) \to 1$ as $n \to \infty$.

Thus, the sequence converges pointwise to the limit function $f\colon [0,1] \to \mathbb{R}$ given by

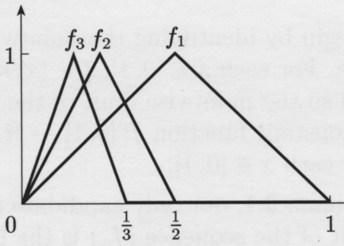
$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

This function is discontinuous at 0. Since

$$\lim_{y \to 0} \frac{\sin y}{y} = 1,$$

f_n is continuous at $x = 0$, for all $n \in \mathbb{N}$. Thus, by Theorem 3.2, the sequence does not converge uniformly.

3.6 (a)



(b) For all $n \in \mathbb{N}$, $f_n(0) = 0$, and so $f_n(0) \to 0$ as $n \to \infty$.

If $0 < x \leq 1$, then $f_n(x) = 0$ for all $n > 1/x$, and hence $f_n(x) \to 0$ as $n \to \infty$.

Thus, for each $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = 0$.

(c) We use the strategy. Step 1 of the strategy requires us to identify a possible limit. By (b), the only possible limit for the sequence (f_n) is the pointwise limit function, that is, the zero function $f\colon [0,1] \to \mathbb{R}$ given by $f(x) = 0$ for all $x \in [0,1]$. This is a continuous function, and so we move to Step 3 of the strategy and calculate $d_{\max}(f_n, f)$, for $n \in \mathbb{N}$.

In fact, since $f_n(\frac{1}{2n}) = 1$ and $f_n(x) < 1$ for $x \neq 1/2n$, we have, for all $n \in \mathbb{N}$,

$$d_{\max}(f_n, f) = \max\{|f_n(x) - 0| : x \in [0,1]\} = 1.$$

Thus the sequence $(d_{\max}(f_n, f))$ is not a null sequence, and so (f_n) does not converge in $C[0,1]$ for $d_{\max}$.

4.1 Let $(X, \mathcal{T})$ be a topological space and let (a_n) be a sequence in X. Suppose that $a_n \to a \in X$. Now let (a_{n_k}) be a subsequence of (a_n). Then $1 \leq n_1 < n_2 < n_3 < \cdots$, and so, in particular, $n_k \geq k$ for all $k \in \mathbb{N}$.

Let U be a neighbourhood of a. We show that the terms of (a_{n_k}) eventually lie in U. Since (a_n) converges to a, there is an $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$. But then, for all $k > N$, $n_k \geq k > N$, and so $a_{n_k} \in U$, as required.

4.2 Since (a_n) is a bounded sequence in $\mathbb{R}$, there is an $M \geq 0$ such that

$$\{a_n : n \in \mathbb{N}\} \subseteq [-M, M].$$

But $[-M, M]$ is a compact set, and so (a_n) is a sequence in a compact subset of $\mathbb{R}$. By Theorem 4.2, (a_n) has a convergent subsequence with limit in $[-M, M]$.

4.3 Let (X, d) be a sequentially compact metric space and let $\varepsilon > 0$. Lemma 4.4 implies that there is a finite ε-net $\{x_1, x_2, \ldots, x_N\}$, for some $\varepsilon > 0$. Let

$$M = \max\{d(x_1, x_2), d(x_1, x_3), \ldots, d(x_1, x_N)\}.$$

Now, by the definition of an ε-net, if $x, y \in X$ there are x_i with $d(x, x_i) < \varepsilon$, and x_j with $d(y, x_j) < \varepsilon$, and so

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 2\varepsilon + M.$$

Hence X is bounded.

4.4 We must show that for each $\varepsilon > 0$, there is a finite ε-net. Let $\varepsilon > 0$.

If $\varepsilon > 1$, then $\{0\}$ is an ε-net for $[0,1]$.

If $0 < \varepsilon \leq 1$, then, for example,

$$\{\tfrac{1}{2}n\varepsilon : n \in \mathbb{N} \text{ and } 0 \leq n \leq \tfrac{2}{\varepsilon}\}$$

is a finite ε-net for $[0,1]$ with the Euclidean metric.

4.5 Let (X, d) be a compact metric space, and let $\varepsilon > 0$. We seek a finite ε-net for X. The collection

$$\{B_d(x, \varepsilon) : x \in X\}$$

is an open cover of X. Since X is compact, the cover has a finite subcover

$$\{B_d(x_1, \varepsilon), B_d(x_2, \varepsilon), \ldots, B_d(x_n, \varepsilon)\}.$$

So $\{x_1, x_2, \ldots, x_n\}$ is a finite ε-net. Since ε is arbitrary, X is totally bounded.

5.1 Let (f_n) be the sequence of functions in A given by

$$f_n(x) = n \quad \text{for each } x \in [0,1].$$

For each $x \in [0,1]$, the sequence $(f_n(x))$ diverges, and so has no pointwise limit; moreover, any subsequence of $(f_n(x))$ also diverges. Lemma 3.1 implies that there is no uniformly convergent subsequence. But, by Theorem 3.3, uniform convergence is the same as $d_{\max}$-convergence in $C[0,1]$. It follows that there are no $d_{\max}$-convergent subsequences of (f_n). Hence, by Theorem 4.2, A is not compact.

5.2 (a) To show that A is compact, Theorem 5.1 tells us that it is sufficient to show that:

(i) A is closed;

(ii) A is a pointwise bounded family of functions;

(iii) A is an equicontinuous family of functions.

(i) *A is closed.*

Let (f_n) be a sequence of functions in A that converges to $f \in C[0,1]$ for $d_{\max}$. We need to show that $f \in A$.

Since $f_n(0) = 0$ for all $n \in \mathbb{N}$,

$$|f(0)| = |f(0) - f_n(0)| \leq d_{\max}(f_n, f).$$

But $d_{\max}(f_n, f) \to 0$ as $n \to \infty$. Hence $f(0) = 0$.

For $x, y \in [0,1]$, the Triangle Inequality implies that, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&|f(x) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq |f_n(x) - f_n(y)| + 2d_{\max}(f_n, f) \\
&\leq \sqrt{|x - y|} + 2d_{\max}(f_n, f),
\end{aligned}$$

since $f_n \in A$. Since $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, it follows that

$$|f(x) - f(y)| \leq \sqrt{|x - y|} \quad \text{for all } x, y \in [0,1].$$

Hence $f \in A$ and so A is closed.

(ii) *A is a pointwise bounded family of functions.*

Let $x \in [0,1]$. Then, for each $f \in A$,

$$|f(x)| = |f(x) - f(0)| \leq \sqrt{|x - 0|} = \sqrt{x} \leq 1.$$

This bound does not depend on the choice of f, and so A is a pointwise bounded family of functions.

(iii) *A is an equicontinuous family of functions.*

Let $\varepsilon > 0$ and fix $a \in [0,1]$. If $x \in [0,1]$ and $|x - a| < \delta$ then, for each $f \in A$,

$$|f(x) - f(a)| \leq \sqrt{|x - a|} < \sqrt{\delta}.$$

Take $\delta = \varepsilon^2$. Then, for each $f \in A$ and each $a \in [0,1]$, $|f(x) - f(a)| < \varepsilon$ whenever $x \in [0,1]$ satisfies $|x - a| < \delta$. Thus A is an equicontinuous family of functions.

Hence A is compact.

(b) F is the composite $G \circ H$ of the continuous functions $H\colon C[0,1] \to C[0,1]$ and $G\colon C[0,1] \to \mathbb{R}$, where $(H(f))(x) = f(x)^2$ and $G(f) = \int_0^1 f(x)\,dx$. Thus F is continuous.

Since F is a continuous function defined on a compact set, the General Extreme Value Theorem (*Unit C2*, Theorem 4.3) implies that F is bounded and attains its bounds on A. In particular, F attains a minimum on A.

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